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Characterizations of Hardy spaces associated to higher order elliptic operators [☆]

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Abstract

In this paper, the authors first show that the classical Hardy space $H^1(\mathbb{R}^n)$ can be characterized by the non-tangential maximal functions and the area integrals associated with the semigroups $e^{-t\mathcal{P}}$ and $e^{-t\sqrt{\mathcal{P}}}$, respectively, where \mathcal{P} is an elliptic operator with real constant coefficients of homogeneous order $2m$ ($m \geq 1$). Moreover, the authors also prove that $H^1(\mathbb{R}^n)$ can be characterized by the Riesz transforms $\nabla^m \mathcal{P}^{-1/2}$ if and only if m is an odd integer. In the main part of this paper, the authors develop a theory of Hardy space associated with L , where L is a higher order divergence form elliptic operator with complex bounded measurable coefficients. The authors set up a molecular Hardy space $H_L^1(\mathbb{R}^n)$ and give its characterizations by area integrals related to the semigroups e^{-tL} and $e^{-t\sqrt{L}}$, respectively. Finally, authors give the (H_L^1, L^1) boundedness of Riesz transforms, square functions and maximal functions associated with L . © 2012 Elsevier Inc. All rights reserved.

Keywords: Higher order elliptic operator; Hardy space; Riesz transform; Square function; Off-diagonal estimates

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1. Introduction

Since the famous works by Stein and Weiss [60] and Fefferman and Stein [34] were published, it was well known that the real Hardy spaces $H^p(\mathbb{R}^n)$ which were extensively and deeply studied not only had a profound effect on the domain of harmonic analysis, but also had a major influence in several areas such as partial differential equations, several complex variables, martingale theory, and analysis on symmetric spaces (see, e.g. [59,35,55,27], etc.).

Historically, the theory of $H^p(\mathbb{R}^n)$ is connected with the solution of Laplace equation on the upper half-spaces \mathbb{R}_+^{n+1} . Explicitly, the elements in $H^p(\mathbb{R}^n)$ can be considered as the boundary value of certain harmonic functions in the upper half-spaces \mathbb{R}_+^{n+1} , or, alternatively in the terms of systems of conjugate harmonic functions which generalized the notion of analytic functions occurring in the case $n = 1$. As real variable structure of Hardy spaces are explored, many

more flexible real approaches to define Hardy spaces were discovered, including ones by very simple maximal function $M_\Phi f$ for some Schwarz function with $\int \Phi = 1$, Riesz transform, the atomic and molecule decompositions and the Littlewood–Paley theory etc. (see [34,14,16,49,61,36,39,63]). In one word, these different characterizations on Hardy spaces provide very powerful insights and great interests in both theories and applications.

Among these Hardy spaces $H^p(\mathbb{R}^n)$ ($0 < p \leq 1$), the only Banach space $H^1(\mathbb{R}^n)$, which plays a special role as a natural substitute of $L^1(\mathbb{R}^n)$, not only has abundant characterization properties, but is also the predual space of famous $BMO(\mathbb{R}^n)$ space introduced originally by John and Nirenberg in [45]. Now let us recall some important characterizations of $H^1(\mathbb{R}^n)$ from the view point of Laplacian and its corresponding Poisson semigroup $\{e^{-t\sqrt{-\Delta}}\}_{t \geq 0}$ and heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$. Explicitly, if we define the following two kinds of area integrals and non-tangential maximal functions:

$$S^{\sqrt{-\Delta}} f(x) := \left(\iint_{\Gamma(x)} |\nabla e^{-t\sqrt{-\Delta}} f(y)|^2 t^{1-n} dy dt \right)^{\frac{1}{2}},$$

$$\mathcal{N}^{\sqrt{-\Delta}} f(x) := \sup_{(y,t) \in \Gamma(x)} |e^{-t\sqrt{-\Delta}} f(y)|$$

and

$$S^{-\Delta} f(x) := \left(\iint_{\Gamma(x)} |\nabla e^{-t^2(-\Delta)} f(y)|^2 t^{1-n} dy dt \right)^{\frac{1}{2}},$$

$$\mathcal{N}^{-\Delta} f(x) := \sup_{(y,t) \in \Gamma(x)} |e^{t\Delta} f(y)|,$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$, then for $f \in H^1(\mathbb{R}^n)$, $\|\mathcal{N}^{\sqrt{-\Delta}} f\|_{L^1(\mathbb{R}^n)}$, $\|S^{\sqrt{-\Delta}} f\|_{L^1(\mathbb{R}^n)}$, $\|\mathcal{N}^{-\Delta} f\|_{L^1(\mathbb{R}^n)}$ and $\|S^{-\Delta} f\|_{L^1(\mathbb{R}^n)}$ are all equivalent norms of the $H^1(\mathbb{R}^n)$, that is

$$\begin{aligned} \|\mathcal{N}^{-\Delta} f\|_{L^1(\mathbb{R}^n)} &\approx \|S^{-\Delta} f\|_{L^1(\mathbb{R}^n)} \approx \|S^{\sqrt{-\Delta}} f\|_{L^1(\mathbb{R}^n)} \approx \|\mathcal{N}^{\sqrt{-\Delta}} f\|_{L^1(\mathbb{R}^n)} \\ &\approx \|f\|_{H^1(\mathbb{R}^n)}. \end{aligned} \quad (1.1)$$

Moreover, $H^1(\mathbb{R}^n)$ also can be characterized by Riesz transforms $\nabla(-\Delta)^{-\frac{1}{2}}$. In fact,

$$H^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : \nabla(-\Delta)^{-\frac{1}{2}} f \in L^1(\mathbb{R}^n) \right\}. \quad (1.2)$$

For more details of characterizations above, one can refer to Fefferman and Stein [34] or Stein's book [59].

In recent years, some kind of new Hardy spaces $H_L^1(\mathbb{R}^n)$ associated with differential operators L were actively studied. As $L = -\Delta$, the new space $H_{-\Delta}^1(\mathbb{R}^n)$ is the same as classical Hardy space $H^1(\mathbb{R}^n)$. However, if the operator L is a general differential operator even with some smooth variable coefficients, then it is not necessarily consistent with the classical one. Such

Hardy space associated to Schrödinger operator $L = -\Delta + V$ with a nonnegative nonzero polynomial potential V was first studied by Dziubański and Zienkiewicz [33], where they defined the $H_L^1(\mathbb{R}^n)$ similarly to classical case by a vertical maximal function associated with Schrödinger semigroup $\{e^{-tL}\}_{t \geq 0}$, as well as showed its characterizations by atoms and square function adapted to Schrödinger operator L . In a subsequent paper [30], Dziubański and Zienkiewicz further investigated $H_L^1(\mathbb{R}^n)$ related to Schrödinger operator with a nonnegative potential V satisfying a reverse Hölder inequality, which contains nonnegative polynomials as typical examples. In this special context of Schrödinger operator, we also refer to [31] and [32] for more recent works related to the specific space $H_L^1(\mathbb{R}^n)$.

On the other hand, motivated partially on the earlier, more specific work of Auscher and Russ [10] on strongly Lipschitz domains Ω of \mathbb{R}^n , only assuming that the kernel of semigroup e^{-tL} generated by an abstract operator L satisfies certain pointwise estimates (here it means that L is not necessarily some type of differential operator), Duong and Yan [28,29] introduced a similar Hardy space $H_L^1(\mathbb{R}^n)$ adapted to the operator L by area integral associated with e^{-tL} , and particularly established its famous new dual space $BMO_L(\mathbb{R}^n)$ which has also several similar important properties as classical $BMO(\mathbb{R}^n)$ space (also see [31] in the context of Schrödinger operator). In view of their general conditions, their results can be applied to several important differential operators classes, typically including Schrödinger operators $-\Delta + V$ mentioned above, and second order elliptic divergence operator $-\operatorname{div}(A(x)\nabla)$, where $A(x)$ is an $n \times n$ matrix defined on \mathbb{R}^n with real L^∞ -coefficients for any dimension n , or complex L^∞ -coefficients for dimension $n = 1, 2$, which satisfies the ellipticity (or “accretivity”) conditions

$$\lambda|\xi|^2 \leq \Re(A\xi \cdot \bar{\xi}) \quad \text{and} \quad |A\xi \cdot \bar{\xi}| \leq \Lambda|\xi||\xi|,$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda < \Lambda < \infty$. Undoubtedly, these works are of fundamental importance to further studies of this subject.

Even so, nevertheless, when L is a second order divergence elliptic operator $-\operatorname{div}(A(x)\nabla)$ with complex L^∞ -coefficients for dimension $n \geq 3$, since its corresponding kernel of semigroup e^{-tL} may fail to have desired Gaussian pointwise estimates (also see Auscher and Tchamitchian [11] for such a specific counterexample as $n \geq 5$), thus the results and methods in [29] cannot be directly applied to this kind of operators. Hence in the recent paper [42], Hofmann and Mayboroda developed a corresponding Hardy space $H_L^1(\mathbb{R}^n)$ in term of proper molecules, and as well obtained its equivalent characterizations by non-tangential maximal function and square function associated with L only using Davies–Gaffney type estimates in place of pointwise kernel bound. In this connection, using such similar type of estimates, Auscher, McIntosh and Russ [7] also considered the Hardy spaces associated to the Hodge–de Rham Laplacian on Riemannian manifold with doubling measure, which further was extended to arbitrary nonnegative, self-adjoint operator L satisfying Davies–Gaffney bounds in general setting of a metric space with a doubling measure by Hofmann et al. [40].

As we have noticed, the papers cited above on the new space $H_L^1(\mathbb{R}^n)$ mainly focus on second order differential operators or the other operators which generate semigroups having kernels satisfying Gaussian pointwise estimates. However, for a uniformly divergence form elliptic operator L with bounded (real or complex) measurable coefficients, if space dimension $n > 2m$, where $2m$ is the order of L , then the kernel of semigroup e^{-tL} may not satisfy Gaussian pointwise estimates, and its semigroup e^{-tL} only has a partial L^p -theory where p belongs to some finite interval (see e.g. Davies [22,23], Auscher [2] and references therein). Hence the methods in [29]

don't apply to this class of higher order divergence form elliptic operators with bounded (real or complex) measurable coefficients.

In the present paper, we attempt to establish a theory of Hardy space $H_L^1(\mathbb{R}^n)$ for such higher order cases.

Up to now, there exist many interesting papers related to higher order elliptic operators and systems with bounded measurable coefficients, see for instance Davies [18–23], Barbatis and Davies [12] for L^p spectral theory and Gaussian estimates of kernel, Auscher, Hofmann, McIntosh and Tchamitchian [6] for the famous higher order Kato square root problem, and Auscher [2] for L^p bound of higher order Riesz transform, etc. Actually, our work on Hardy space $H_L^1(\mathbb{R}^n)$ connected with higher order operators is partially motivated by these studies. In this paper, the authors define Hardy space $H_L^1(\mathbb{R}^n)$ in molecules form adapted to divergence form elliptic operators of order $2m \geq 4$ with bounded measurable coefficients and establish its characterizations by the area integrals associated with semigroups e^{-tL} and $e^{-t\sqrt{L}}$. Our methods mainly depend on the analytic properties and higher order type off-diagonal estimates of semigroup e^{-tL} on $L^p(\mathbb{R}^n)$ (see Theorem 3.2 below for the details). Although our characterizations on $H_L^1(\mathbb{R}^n)$ share a lot of similar ideas with the second order theory developed by Hofmann and Mayboroda [42], however, we remark that other important characterizations such as Riesz transforms $\nabla^m L^{-1/2}$ may cause failure even in the cases of higher order operators with smooth coefficients.

In fact, in Section 2 we first discuss the homogeneous elliptic operator \mathcal{P} of order $2m$ with real constant coefficients, which is defined by

$$\mathcal{P} := \sum_{|\alpha|=2m} a_\alpha \mathcal{D}^\alpha, \quad (1.3)$$

where $m \in \mathbb{N}$, $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n)$, $\mathcal{D}_j = \frac{1}{i} \frac{\partial}{\partial x_j}$ and α is a multi-index, that is, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$.

On the one hand, we will see that if all coefficients of the homogeneous polynomial $P(\xi)$, the symbol of operator \mathcal{P} , are real and $P(\xi) > 0$ for all $\xi \neq 0$, then the classical Hardy space $H^1(\mathbb{R}^n)$ can be characterized by the non-tangential maximal functions $\mathcal{N}^\mathcal{P}$ (or $\mathcal{N}^{\sqrt{\mathcal{P}}}$), the area integrals $\mathcal{S}^\mathcal{P}$ (or $\mathcal{S}^{\sqrt{\mathcal{P}}}$) and the vertical square functions $g^\mathcal{P}$ (or $g^{\sqrt{\mathcal{P}}}$) associated with the heat semigroup $e^{-t\mathcal{P}}$ (or Poisson type semigroup $e^{-t\sqrt{\mathcal{P}}}$). In particular, the above results show that the characterizations of $H^1(\mathbb{R}^n)$ given by the non-tangential maximal functions, the area integrals and the vertical square functions are all same if we replace the Laplacian $-\Delta$ by the polyharmonic operator $(-\Delta)^m$ for all $m \geq 2$, respectively.

On the other hand, inspired by (1.2), it is very natural to ask whether $H^1(\mathbb{R}^n)$ also can be characterized by the higher order Riesz transforms $\nabla^m (-\Delta)^{-m/2}$ ($m \geq 2$), where $\nabla^m (-\Delta)^{-m/2}$ is defined by

$$\nabla^m ((-\Delta)^m)^{-1/2} f = (\mathcal{R}^\alpha f)_{|\alpha|=m},$$

where $\alpha \in \mathbb{Z}_+^n$ and $\mathcal{R}^\alpha = R_1^{\alpha_1} \circ \dots \circ R_n^{\alpha_n}$ with

$$R_j^{\alpha_j} = \underbrace{\partial_j (-\Delta)^{-1/2} \circ \dots \circ \partial_j (-\Delta)^{-1/2}}_{\alpha_j} \quad (j = 1, 2, \dots, n).$$

However, we find that there exists an essential difference between $\nabla(-\Delta)^{-1/2}$ and $\nabla^m(-\Delta)^{-m/2}$ ($m \geq 2$) in characterizing $H^1(\mathbb{R}^n)$. In fact, for a general homogeneous elliptic operator \mathcal{P} defined by (1.3), by using a famous result of Uchiyama [62] (also see remarks of Stein [59, p. 184]), we prove that $H^1(\mathbb{R}^n)$ can be characterized by $\nabla^m \mathcal{P}^{-1/2}$, the Riesz transforms associated with \mathcal{P} , if and only if m is an odd integer (see Theorem 2.3 in Section 2.3). Particularly, if $P(\xi) = |\xi|^2$ (in this case, $m = 1$ and $\mathcal{P} = -\Delta$), we get (1.2) again. When $P(\xi) = |\xi|^{2m}$, $m \geq 2$, we have $\nabla^m \mathcal{P}^{-1/2} = \nabla^m ((-\Delta)^m)^{-1/2}$, so $H^1(\mathbb{R}^n)$ can be characterized by the higher order Riesz transforms $\nabla^m ((-\Delta)^m)^{-1/2}$ if and only if m is an odd integer. Anyway, it should be emphasized that the result above displays some distinct phenomenons in the context of higher order elliptic operators (even the polyharmonic operator $(-\Delta)^m$).

For the remaining parts of this paper, we aim at the Hardy space $H_L^1(\mathbb{R}^n)$ adapted to the following homogeneous higher order elliptic operator L of order $2m$ ($m \geq 2$) in divergence form

$$Lf := (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha,\beta} \partial^\beta f) \quad (1.4)$$

with complex coefficients $a_{\alpha,\beta} \in L^\infty(\mathbb{R}^n, \mathbb{C})$ for all multi-indices α, β satisfying $|\alpha| = |\beta| = m$, which is interpreted by the following sesquilinear form:

$$\mathcal{Q}(f, g) := \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha,\beta}(x) \partial^\beta f(x) \partial^\alpha \bar{g}(x) dx,$$

satisfies that

$$|\mathcal{Q}(f, g)| \leq \Lambda \|\nabla^m f\|_{L^2} \|\nabla^m g\|_{L^2} \quad (1.5)$$

and the strong Gårding inequality

$$\Re(\mathcal{Q}(f, f)) \geq \lambda \|\nabla^m f\|_{L^2}^2 \quad (1.6)$$

for some $\lambda > 0$ and $\Lambda < \infty$ independent of $f, g \in W^{m,2}$, the Sobolev space. In addition, here and in the sequel, $\partial^\alpha = \partial^{\alpha_1} / \partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}$. It was well known that such a higher order operator L generates a C_0 -semigroup $\{e^{-tL}\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$ which can be extended into an analytic semigroup $\{e^{-zL}\}_{z \in \sum_{j=1}^n \frac{\pi}{2} - \omega}$ of $L^2(\mathbb{R}^n)$ for some $\omega \in [0, \frac{\pi}{2})$ and also permits a bounded holomorphic functional calculus on L^2 (see e.g. Kato [47, p. 492], McIntosh [52], McIntosh and Yagi [53], and Auscher [2]). Moreover, it is remarkable that the semigroup $\{e^{-tL}\}_{t \geq 0}$ always satisfies Davies–Gaffney estimates of higher order type on $L^2(\mathbb{R}^n)$ for any dimension n (see Theorem 3.2 in Section 3), which are essential to our studies here. For more details about the estimates of the semigroup $\{e^{-tL}\}_{t \geq 0}$, one can see Section 3 of this paper.

The paper is organized as follows:

In Section 2, we give the characterizations of the classical Hardy space $H^1(\mathbb{R}^n)$ through the maximal functions, area integrals, the square functions and Riesz transforms associated with operator \mathcal{P} . In Section 3, we introduce some known results and useful lemmas about the off-diagonal estimates of some families of operators and the L^p boundedness of the square functions related to the operator L . The definition of the molecules and molecular Hardy spaces $H_L^1(\mathbb{R}^n)$

are given in Section 4. In Sections 5 and 6, we give the characterizations of Hardy space $H_L^1(\mathbb{R}^n)$ by the area integrals associated with semigroups e^{-tL} and $e^{-t\sqrt{L}}$, respectively. In Section 7, as some applications, we show that vertical square functions, the Riesz transforms and vertical maximal functions associated with the operator L are all bounded from H_L^1 to L^1 . Finally, Section 8 sums up many other possible characterizations of the Hardy space $H_L^1(\mathbb{R}^n)$ and some further interesting works related to higher order Schrödinger type operator $(-\Delta)^m + V$.

Remark 1.1. In [25], we develop a theory of $BMO_L(\mathbb{R}^n)$ and $VMO_L(\mathbb{R}^n)$ associated with the higher order divergence form elliptic operator L defined by (1.4)–(1.6) and give the corresponding dual space and predual space of $H_L^1(\mathbb{R}^n)$. On the other hand, when L is a second order operator, the Hardy spaces $H_L^p(\mathbb{R}^n)$ ($0 < p < \infty$) and their applications have been studied in [43] and [44], respectively. The investigations of the Hardy spaces $H_L^p(\mathbb{R}^n)$ ($0 < p < \infty$) related to the higher order divergence form elliptic operator L will be given in our forthcoming paper.

2. Characterizations of $H^1(\mathbb{R}^n)$ associated with operator \mathcal{P}

In this section, we always assume that \mathcal{P} is a homogeneous elliptic operator of order $2m$ defined by (1.3) and that its symbol $P(\xi)$ has real constant coefficients and $P(\xi) > 0$ for all $\xi \neq 0$.

As we mentioned in Section 1, the classical Hardy space $H^1(\mathbb{R}^n)$ can be characterized via Laplacian (see (1.1)). In this section, we will show that this fact also holds when replacing $-\Delta$ by \mathcal{P} .

In the first two subsections of Section 2, we give the definitions of the non-tangential maximal functions and the area integrals associated with the semigroups $\{e^{-s\mathcal{P}}\}_{s \geq 0}$ and $\{e^{-s\sqrt{\mathcal{P}}}\}_{s \geq 0}$, respectively. One can see that the Hardy spaces defined via the non-tangential maximal function, the area integral and the vertical square function associated with \mathcal{P} are all equal to $H^1(\mathbb{R}^n)$.

However, in the final subsection, we define a Hardy space $H_{\nabla^m \mathcal{P}^{-1/2}}^1(\mathbb{R}^n)$ by the Riesz transforms $\nabla^m \mathcal{P}^{-1/2}$. We will prove that $H_{\nabla^m \mathcal{P}^{-1/2}}^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ if and only if m is an odd integer.

In the following discussion, replacing $\{e^{-s\mathcal{P}}\}$ and $\{e^{-s\sqrt{\mathcal{P}}}\}$, we will consider the family of operators $\{e^{-t^{2m}\mathcal{P}}\}$ and $\{e^{-t^m\sqrt{\mathcal{P}}}\}$ respectively, which can be obtained by changing variables.

2.1. Characterizations of H^1 by the maximal function $\mathcal{N}^{\mathcal{P}}$, the area integral $\mathcal{S}^{\mathcal{P}}$ and the vertical square function $g^{\mathcal{P}}$

First, we define the non-tangential maximal function $\mathcal{N}^{\mathcal{P}}$, the area integral $\mathcal{S}^{\mathcal{P}}$ and the square function $g^{\mathcal{P}}$ associated with the family of operators $\{e^{-t^{2m}\mathcal{P}}\}_{t \geq 0}$ respectively by

$$\mathcal{N}^{\mathcal{P}} f(x) = \sup_{(y,t) \in \Gamma(x)} |e^{-t^{2m}\mathcal{P}} f(y)|,$$

$$g^{\mathcal{P}} f(x) = \left(\int_0^\infty |t^{2m} \mathcal{P} e^{-t^{2m}\mathcal{P}} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$\mathcal{S}^{\mathcal{P}} f(x) = \left(\iint_{\Gamma(x)} |t^{2m} \mathcal{P} e^{-t^{2m} \mathcal{P}} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

where and in the sequel, $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1}: |x - y| < t\}$ for $t > 0$. Then we have the following results:

Theorem 2.1. For $m \geq 1$, we have

- (i) $H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): \mathcal{N}^{\mathcal{P}} f \in L^1(\mathbb{R}^n)\}$ and $\|f\|_{H^1(\mathbb{R}^n)} \approx \|\mathcal{N}^{\mathcal{P}} f\|_{L^1(\mathbb{R}^n)}$.
- (ii) $H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): \mathcal{S}^{\mathcal{P}} f \in L^1(\mathbb{R}^n)\}$ and $\|f\|_{H^1(\mathbb{R}^n)} \approx \|\mathcal{S}^{\mathcal{P}} f\|_{L^1(\mathbb{R}^n)}$.
- (iii) $H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): g^{\mathcal{P}} f \in L^1(\mathbb{R}^n)\}$ and $\|f\|_{H^1(\mathbb{R}^n)} \approx \|g^{\mathcal{P}} f\|_{L^1(\mathbb{R}^n)}$.

Proof. (i) Observe that

$$e^{-t^{2m} \mathcal{P}} f(x) = \Phi_t * f(x)$$

with $\Phi(x) = \mathcal{F}^{-1}(e^{-P(\cdot)})(x)$, where and in the sequel, \mathcal{F} and “ \wedge ” denote the Fourier transform and \mathcal{F}^{-1} denotes the Fourier inverse transform. Then we have that $\Phi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz function space) and $\int_{\mathbb{R}^n} \Phi(x) dx = 1$. Then by [59, Theorem 1, p. 91], we finish the proof of (i).

The proof of (ii) and (iii): Notice that

$$t^{2m} \mathcal{P} e^{-t^{2m} \mathcal{P}} f(x) = \Psi_t * f(x)$$

with $\Psi(x) = \mathcal{F}^{-1}(P(\cdot)e^{-P(\cdot)})(x)$. Thus we still have that $\Psi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \Psi(x) dx = 0$ since $P(0) = 0$. Moreover, Ψ is nondegenerate (see [59, p. 186] for the definition). Then by [35, Theorem 7.8, p. 225] and [35, Corollary 7.27, p. 244], we conclude the proof of (ii). Similarly, we could use [35, Theorem 7.7, p. 223] and [35, Corollary 7.28, p. 244] to get (iii). \square

Remark 2.1. In [29] and [28], Duong and Yan introduced the bounded mean oscillation spaces $BMO_L(\mathbb{R}^n)$ associated with an operator L . They showed that if the kernel of the semigroup e^{-tL} generated by L satisfies some pointwise estimates, then $(H_L^1)^* = BMO_{L^*}$ where $H_L^1(\mathbb{R}^n)$ denotes the Hardy space associated to L , and L^* is the adjoint operator of L . We refer the reader to [29] and [28] for the definition of BMO_{L^*} and more details on the above facts. Now we denote

$$H_{\mathcal{P}}^1(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n): \mathcal{S}^{\mathcal{P}} f \in L^1(\mathbb{R}^n)\}.$$

Note that $\mathcal{P}^* = \mathcal{P}$, thus from Theorem 2.1 and [29, Theorem 3.1], we immediately get

$$BMO_{\mathcal{P}}(\mathbb{R}^n) = (H_{\mathcal{P}}^1(\mathbb{R}^n))^* = (H^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n). \quad (2.1)$$

2.2. Characterizations of H^1 by the maximal function $\mathcal{N}^{\sqrt{\mathcal{P}}}$, the area integral $\mathcal{S}^{\sqrt{\mathcal{P}}}$ and the vertical square function $g^{\sqrt{\mathcal{P}}}$

Before proving the characterizations of H^1 , we give some estimates related to the family of operators $e^{-t\sqrt{\mathcal{P}}}$, which play a key role in the proof of Theorem 2.2 below. Let $P(\xi)$ be a

homogeneous elliptic polynomial of order $2m$ ($m \geq 1$) on \mathbb{R}^n with real constant coefficients and $P(\xi) > 0$ for all $\xi \neq 0$. For $\ell = 0, 1, 2, \dots$, denote

$$K^\ell(x) = \mathcal{F}^{-1}(P(\cdot)^\ell e^{-\sqrt{P(\cdot)}})(x),$$

and

$$G^\ell(x) = \mathcal{F}^{-1}(P(\cdot)^{\ell+\frac{1}{2}} e^{-\sqrt{P(\cdot)}})(x).$$

Lemma 2.1. For $\alpha \in \mathbb{N}^n \cup \{0\}$, $\partial^\alpha K^\ell$ and $\partial^\alpha G^\ell$ are both smooth functions on \mathbb{R}^n and there exists constant $C_{\ell,\alpha}$ such that

$$|\partial^\alpha K^\ell(x)| \leq C_{\ell,\alpha} (1 + |x|)^{-n-|\alpha|-(2\ell+1)m} \quad (2.2)$$

and

$$|\partial^\alpha G^\ell(x)| \leq C_{\ell,\alpha} (1 + |x|)^{-n-|\alpha|-(2\ell+1)m}. \quad (2.3)$$

Proof. We first consider (2.2). Let $\sigma(\xi)$ be a $C_c^\infty(\mathbb{R}^n)$ function satisfying $\sigma(\xi) = 1$ for $|\xi| \leq 1$ and $\sigma(\xi) = 0$ for $|\xi| \geq 2$. We have

$$\begin{aligned} K^\ell(x) &= \mathcal{F}^{-1}(\sigma(\cdot)P(\cdot)^\ell e^{-\sqrt{P(\cdot)}})(x) + \mathcal{F}^{-1}((1 - \sigma(\cdot))P(\cdot)^\ell e^{-\sqrt{P(\cdot)}})(x) \\ &:= K_1^\ell(x) + K_2^\ell(x). \end{aligned}$$

Since $K_2^\ell \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), it suffices to show

$$|\partial^\alpha K_1^\ell(x)| \leq C_{\ell,\alpha} (1 + |x|)^{-n-|\alpha|-(2\ell+1)m}. \quad (2.4)$$

To estimate (2.4), we write

$$\begin{aligned} K_1^\ell(x) &= \mathcal{F}^{-1}(\sigma(\cdot)P(\cdot)^\ell)(x) + \mathcal{F}^{-1}(\sigma(\cdot)P(\cdot)^\ell(e^{-\sqrt{P(\cdot)}} - 1))(x) \\ &:= K_{11}^\ell(x) + K_{12}^\ell(x). \end{aligned}$$

Notice that $K_{11}^\ell \in \mathcal{S}(\mathbb{R}^n)$, which implies that we only need to show

$$|\partial^\alpha K_{12}^\ell(x)| \leq C_{\ell,\alpha} (1 + |x|)^{-n-|\alpha|-(2\ell+1)m}. \quad (2.5)$$

To this end, we denote

$$\omega(\xi) = \sigma(\xi)P(\xi)^\ell(e^{-\sqrt{P(\xi)}} - 1).$$

Then by the Leibniz formula, we have for all $\xi \in \text{supp}(\sigma)$ and $v \in \mathbb{N}^n$

$$\begin{aligned} |\partial^v (P(\xi)^\ell (e^{-\sqrt{P(\xi)}} - 1))| &= \left| \sum_{v_1+v_2=v} C_v^{v_1} \partial^{v_1} (P(\xi)^\ell) \partial^{v_2} (e^{-\sqrt{P(\xi)}} - 1) \right| \\ &\leq \sum_{v_1+v_2=v} C_{\ell,v} |\xi|^{2m\ell-|v_1|} |\xi|^{m-|v_2|} \\ &\leq C_{\ell,v} |\xi|^{m(2\ell+1)-|v|}, \end{aligned}$$

where we use the fact that $|e^{-\sqrt{P(\xi)}} - 1| \leq C|\xi|^m$ and the estimate $\partial^{v_2} (e^{-\sqrt{P(\xi)}}) \leq c_{v_2} |\xi|^{m-|v_2|}$ for $\xi \in \text{supp}(\sigma)$. Hence by using Leibniz's formula again, we obtain

$$|\partial^\mu \omega(\xi)| \leq \sum_{\mu_1+\mu_2=\mu} C_\mu^{\mu_1} |\partial^{\mu_1} \sigma(\xi) \partial^{\mu_2} (P(\xi)^\ell (e^{-\sqrt{P(\xi)}} - 1))| \leq C_{\ell,\mu} |\xi|^{m(2\ell+1)-|\mu|}.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2\}$ and $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for $\xi \neq 0$. Then we have

$$\omega(\xi) = \sum_{j \in \mathbb{Z}} \omega(\xi) \varphi(2^{-j}\xi) := \sum_{j \in \mathbb{Z}} \omega_j(\xi).$$

Let $K_{12,j}^\ell(x) = \mathcal{F}^{-1}(\omega_j(\cdot))(x)$, then by the dominate convergence theorem, we only need to estimate $\sum_{j \in \mathbb{Z}} |\partial^\alpha K_{12,j}^\ell(x)|$. For each ω_j ($j \in \mathbb{Z}$) and any $N \in \mathbb{N} \cup \{0\}$, by integration by parts, we have

$$\begin{aligned} |\partial^\alpha K_{12,j}^\ell(x)| &= \left| \int_{\mathbb{R}^n} e^{ix\xi} (i\xi)^\alpha \omega_j(\xi) d\xi \right| \\ &\leq C_N |x|^{-N} \int_{\mathbb{R}^n} \sum_{\beta_1+\beta_2+\beta_3=N} |\partial^{\beta_1} \xi^\alpha \partial^{\beta_2} \omega(\xi) \partial^{\beta_3} (\varphi(2^{-j}\xi))| d\xi \\ &\leq C_{\ell,\alpha,N} |x|^{-N} 2^{j(2m\ell+m+n+|\alpha|-N)}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |\partial^\alpha K_{12,j}^\ell(x)| &\leq \sum_{2^j|x| \leq 1} |\partial^\alpha K_{12,j}^\ell(x)| + \sum_{2^j|x| > 1} |\partial^\alpha K_{12,j}^\ell(x)| \\ &\leq C_{\ell,\alpha} \sum_{2^j|x| \leq 1} 2^{j(2m\ell+m+n+|\alpha|)} + C_{\ell,\alpha,N} \sum_{2^j|x| > 1} |x|^{-N} 2^{j(2m\ell+m+n+|\alpha|-N)} \\ &\leq C_{\ell,\alpha} |x|^{-n-|\alpha|-(2\ell+1)m}, \end{aligned}$$

where we choose $N = 0$ and $N > 2m\ell + m + n + |\alpha|$ in the second equality for the first and the second terms, respectively. Hence we finish the proof of (2.2).

The proof of (2.3) is quite similar to (2.2). Note that $(1 - \sigma(\xi))P(\xi)^{\ell+1/2}e^{-\sqrt{P(\xi)}} \in \mathcal{S}(\mathbb{R}^n)$ if σ is defined as above, we only need to give a corresponding estimate (2.4) for G^ℓ . To this end, let $\tau(\xi) = \sigma(\xi)(P(\xi))^{\ell+\frac{1}{2}}e^{-\sqrt{P(\xi)}}$, then by Leibniz's formula, we have

$$|\partial^\mu \tau(\xi)| \leq C_{\ell,\mu} |\xi|^{m(2\ell+1)-|\mu|}.$$

Now, taking φ as above and using the same method, for any $N \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} e^{ix\xi} (i\xi)^\alpha \tau(\xi) \varphi(2^{-j}\xi) d\xi \right| &\leq C_N |x|^{-N} \int_{\mathbb{R}^n} \sum_{\beta_1+\beta_2+\beta_3=N} |\partial^{\beta_1} \xi^\alpha \partial^{\beta_2} \tau(\xi) \partial^{\beta_3} (\varphi(2^{-j}\xi))| d\xi \\ &\leq C_{\ell,\alpha,N} |x|^{-N} 2^{j(2m\ell+m+n+|\alpha|-N)}. \end{aligned}$$

The rest of the proof is exactly the same as the proof of (2.2), we omit the details here. Hence we finish the proof of Lemma 2.1. \square

Remark 2.2. Lemma 2.1 is interesting in its own right. Here we give some remarks as follows.

(i) Comparing the estimates (2.2) with (2.3) in Lemma 2.1, the main difference is due to the fact that the regularity factor $P(\xi)^l$ inside the function $K^l(x)$ is always smooth for any $P(\xi)$, and the factor $P(\xi)^{l+\frac{1}{2}}$ of G^l is possibly singular at 0 for some positive elliptic polynomial $P(\xi)$ (e.g. $P(\xi) = |\xi|^6$ corresponding to the cube-Laplace operator $(-\Delta)^3$).

(ii) As a special case, if $P(\xi) = |\xi|^2$ (i.e. the symbol of the Laplacian $-\Delta$), then it follows by scaling and Lemma 2.1 that the kernels $K_j(t, x)$ of the family of operators $(\sqrt{-\Delta})^j e^{-t\sqrt{-\Delta}}$ ($j = 0, 1, 2, \dots$) have the following pointwise estimates: for $t > 0$

$$|K_j(t, x)| \leq \begin{cases} C_j t^{-n-j} (1+t^{-1}|x|)^{-n-j-1}, & j \text{ is even,} \\ C_j t^{-n-j} (1+t^{-1}|x|)^{-n-j}, & j \text{ is odd.} \end{cases}$$

Obviously, the decay estimates of the $K_j(t, x)$ for the even integers j are better than the odd cases (see Remark 2.2(i) above for the reasons). Indeed, such the different estimates also can be verified by directly checking the following exact formula of Poisson kernel (see e.g. [59, p. 24])

$$K_0(t, x) = \Gamma\left(\frac{n+1}{2}\right) \pi^{-(n+1)/2} \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}$$

and its derivatives $K_j(t, x) = (\frac{d^j}{dt^j} K_0)(t, x)$ for $j = 1, 2, \dots$.

(iii) Let $\phi(D)^\nu e^{-\phi(D)^\mu}$ be a pseudo-differential operator associated with symbol $\phi(\xi)^\nu e^{-\phi(\xi)^\mu}$, where D is defined by (1.3), $\mu > 0$, $\nu > -n$ and $\phi(\xi)$ is a positive smooth homogeneous function of degree one on $\mathbb{R}^n \setminus \{0\}$. By using the same method of proving Lemma 2.1, we also have the following pointwise estimates for their kernels:

$$|\mathcal{F}^{-1}(e^{-\phi(\cdot)^\mu})(x)| \leq C(1+|x|)^{-n-\mu},$$

and for $0 \neq \nu > -n$,

$$|\mathcal{F}^{-1}(\phi(\cdot)^\nu e^{-\phi(\cdot)^\mu})(x)| \leq C(1+|x|)^{-n-\nu}.$$

When $\phi(\xi) = |\xi|$, i.e. the corresponding operator $\phi(\mathcal{D}) = \sqrt{-\Delta}$, the authors in [54] also got the such pointwise estimate of the kernel of $e^{-(\cdot)\mu}$ with $\mu > 0$.

We now give some definitions. The non-tangential maximal function $\mathcal{N}^{\sqrt{\mathcal{P}}}$, the area integral $\mathcal{S}^{\sqrt{\mathcal{P}}}$ and the vertical square function $g^{\sqrt{\mathcal{P}}}$ associated with the family of operators $\{e^{-t^m\sqrt{\mathcal{P}}}\}_{t \geq 0}$ are defined by

$$\mathcal{N}^{\sqrt{\mathcal{P}}} f(x) = \sup_{(y,t) \in \Gamma(x)} |e^{-t^m\sqrt{\mathcal{P}}} f(y)|,$$

$$g^{\sqrt{\mathcal{P}}} f(x) = \left(\int_0^\infty |t^m \sqrt{\mathcal{P}} e^{-t^m\sqrt{\mathcal{P}}} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

and

$$\mathcal{S}^{\sqrt{\mathcal{P}}} f(x) = \left(\iint_{\Gamma(x)} |t^m \sqrt{\mathcal{P}} e^{-t^m\sqrt{\mathcal{P}}} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

respectively. Then the following conclusions hold:

Theorem 2.2. For $m \geq 3$, we have

- (i) $H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): \mathcal{N}^{\sqrt{\mathcal{P}}} f \in L^1(\mathbb{R}^n)\}$ and $\|f\|_{H^1(\mathbb{R}^n)} \approx \|\mathcal{N}^{\sqrt{\mathcal{P}}} f\|_{L^1(\mathbb{R}^n)}$.
- (ii) $H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): \mathcal{S}^{\sqrt{\mathcal{P}}} f \in L^1(\mathbb{R}^n)\}$ and $\|f\|_{H^1(\mathbb{R}^n)} \approx \|\mathcal{S}^{\sqrt{\mathcal{P}}} f\|_{L^1(\mathbb{R}^n)}$.
- (iii) $H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): g^{\sqrt{\mathcal{P}}} f \in L^1(\mathbb{R}^n)\}$ and $\|f\|_{H^1(\mathbb{R}^n)} \approx \|g^{\sqrt{\mathcal{P}}} f\|_{L^1(\mathbb{R}^n)}$.

Proof. We first prove (ii) and (iii). Denote

$$(t^m \sqrt{\mathcal{P}} e^{-t^m\sqrt{\mathcal{P}}} f)(x) = (\Psi_t * f)(x),$$

where $\Psi(x) = \mathcal{F}^{-1}(\sqrt{P(\cdot)} e^{-\sqrt{P(\cdot)}})(x)$. Applying (2.3) (with $\ell = \alpha = 0$ and $\ell = 0$, $\alpha = 1$, respectively), we have that $|\Psi(x)| \leq C(1 + |x|)^{-(n+m)}$ and $|\nabla \Psi(x)| \leq C(1 + |x|)^{-(n+m+1)}$, respectively. Moreover, it is easy to check that $\int_{\mathbb{R}^n} \Psi(x) dx = 0$ and Ψ is nondegenerate. Then by using the theorem in [46, p. 80], we get (ii) and (iii).

Now let us turn to (i). Observe that

$$(e^{-t^m\sqrt{\mathcal{P}}} f)(x) = (\Phi_t * f)(x) \tag{2.6}$$

with $\Phi(x) = \mathcal{F}^{-1}(e^{-\sqrt{P(\cdot)}})(x)$. Then $\int_{\mathbb{R}^n} \Phi(x) dx = 1$ and $|\Phi(x)| \leq C(1 + |x|)^{-(n+m)}$ and $|\nabla \Phi(x)| \leq C(1 + |x|)^{-(n+m+1)}$ by (2.2). By [59, p. 130] it holds that

$$\|\mathcal{N}^{\sqrt{\mathcal{P}}} f\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{H^1(\mathbb{R}^n)}$$

for $f \in H^1(\mathbb{R}^n)$. On the other hand, we choose φ to be a continuous and integrable function on $[1, \infty)$ satisfying $|\varphi(s)| \leq C_N s^{-N}$ for all $N > 0$ and

$$\int_1^\infty s^k \varphi(s) ds = \begin{cases} 1, & k=0; \\ 0, & k=1, 2, \dots \end{cases}$$

We now set

$$\sigma(x) = \int_1^\infty \varphi(s) \Phi_s(x) ds,$$

where $\Phi(x) = \mathcal{F}^{-1}(e^{-\sqrt{P(\cdot)}})(x)$ as before. Then the Fourier transform of σ is

$$\hat{\sigma}(\xi) = \int_1^\infty \varphi(s) e^{-\sqrt{P(s\xi)}} ds.$$

It is easy to verify that $\hat{\sigma}$ is in the Schwartz class (see [34] or [50, p. 8]). Thus, for $f \in H^1(\mathbb{R}^n)$, we have

$$\begin{aligned} \|f\|_{H^1(\mathbb{R}^n)} &= \left\| \sup_{t>0} |\sigma_t * f| \right\|_{L^1(\mathbb{R}^n)} \\ &= \left\| \sup_{t>0} \left| \int_1^\infty \varphi(s) \Phi_{ts} * f(x) ds \right| \right\|_{L^1(\mathbb{R}^n)} \\ &\leq \left\| \int_1^\infty |\varphi(s)| \sup_{t>0} |\Phi_{ts} * f(x)| ds \right\|_{L^1(\mathbb{R}^n)} \\ &\leq C \|\mathcal{N}^{\sqrt{P}} f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

We hence finish the proof of conclusion (i) of Theorem 2.2. \square

Remark 2.3. Similar to (2.1), we have

$$BMO_{\sqrt{P}}(\mathbb{R}^n) = (H^1_{\sqrt{P}}(\mathbb{R}^n))^* = (H^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n), \quad (2.7)$$

where

$$H^1_{\sqrt{P}}(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n) : S^{\sqrt{P}} f \in L^1(\mathbb{R}^n)\}.$$

2.3. Characterizations of H^1 by the Riesz transforms $\nabla^m \mathcal{P}^{-1/2}$

It is well known that the classical Hardy space $H^1(\mathbb{R}^n)$ can be characterized by the classical Riesz transforms $\nabla(-\Delta)^{-1/2}$ (see (1.2)). In this subsection, we will see that $H^1(\mathbb{R}^n)$ can be characterized by the Riesz transforms $\nabla^m \mathcal{P}^{-1/2}$ if and only if m is an odd integer. We define the Hardy space $H_{\nabla^m \mathcal{P}^{-1/2}}^1(\mathbb{R}^n)$ as follows:

$$H_{\nabla^m \mathcal{P}^{-1/2}}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n): \nabla^m \mathcal{P}^{-1/2} f \in L^1(\mathbb{R}^n)\},$$

where $\nabla^m f = (\partial^\alpha f)_{|\alpha|=m}$ is a vector in \mathbb{C}^d with $d = C_{m+n-1}^{n-1}$. Then we have the following result:

Theorem 2.3. *For $m \geq 1$, $H_{\nabla^m \mathcal{P}^{-1/2}}^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ if and only if m is an odd integer. In this case, we have*

$$\|\nabla^m \mathcal{P}^{-1/2} f\|_{L^1(\mathbb{R}^n)} \approx \|f\|_{H^1(\mathbb{R}^n)}.$$

Proof. Write $\nabla^m \mathcal{P}^{-1/2} f = (\partial^\alpha \mathcal{P}^{-1/2} f)_{|\alpha|=m}$ and denote

$$\left((\partial^\alpha \mathcal{P}^{-1/2} f)^\wedge(\xi) \right)_{|\alpha|=m} = \left((-i)^m \frac{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}}{P(\xi)^{1/2}} \hat{f}(\xi) \right)_{|\alpha|=m}. \quad (2.8)$$

Obviously, for any multi-index α with $|\alpha| = m$, then

$$\theta_\alpha(\xi) := (-i)^m \frac{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}}{P(\xi)^{1/2}}$$

is a bounded function of homogeneous of degree zero and C^∞ away from origin on \mathbb{R}^n . Denote

$$\left((-i)^m \frac{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}}{P(\xi)^{1/2}} \right)_{|\alpha|=m} = (\tilde{\theta}_1, \tilde{\theta}_2, \dots, \tilde{\theta}_d),$$

where $d = C_{m+n-1}^{n-1}$. Then, it is obvious that for any $\xi \neq 0$

$$\text{rank} \begin{pmatrix} 1, \tilde{\theta}_1(\xi), & \dots, & \tilde{\theta}_d(\xi) \\ 1, \tilde{\theta}_1(-\xi), & \dots, & \tilde{\theta}_d(-\xi) \end{pmatrix} \equiv 2 \quad (2.9)$$

if and only if m is an odd integer. Let

$$T_{\theta_\alpha} f(x) = (\theta_\alpha(\xi/|\xi|) \hat{f}(\xi))^\vee(x) \quad \text{for } |\alpha| = m. \quad (2.10)$$

Thus, by (2.9), a well-known fact (see [62, p. 218] or [59, p. 184]) tells us that the vector of operators $\{\mathbf{I}, T_{\theta_\alpha}\}_{|\alpha|=m}$ of the singular integrals defined by (2.10) characterizes $H^1(\mathbb{R}^n)$, where and in the sequel \mathbf{I} is the identity operator. In other words, we indeed prove that

$$H^1_{\nabla^m \mathcal{P}^{-1/2}}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$$

if and only if m is an odd integer. \square

Remark 2.4. Note that $(\nabla^m \mathcal{P}^{-1/2})^* = (-1)^m \nabla^m \mathcal{P}^{-1/2}$, then by Theorem 2.3, we immediately get the following conclusion:

$$BMO_{(\nabla^m \mathcal{P}^{-1/2})^*}(\mathbb{R}^n) = BMO(\mathbb{R}^n)$$

if and only if m is an odd integer, where $BMO_{(\nabla^m \mathcal{P}^{-1/2})^*}(\mathbb{R}^n)$ denotes the dual space of $H^1_{\nabla^m \mathcal{P}^{-1/2}}(\mathbb{R}^n)$.

Remark 2.5. If m is an even integer, then $H^1(\mathbb{R}^n) \subsetneq H^1_{\nabla^m \mathcal{P}^{-1/2}}(\mathbb{R}^n)$ by the classical Calderón–Zygmund singular integral theory.

3. L^p – L^q estimates

In this section, we will give some L^p – L^q off-diagonal estimates related to semigroup $\{e^{-tL}\}$ (i.e. Theorem 3.2 below) and the L^p boundedness of the square functions $g_h^{L,k}$ and $\mathcal{S}_h^{L,k}$ associated with the operator L , which play an important role in the proofs of the main results of this paper.

3.1. Off-diagonal estimates for general families of operators

Definition 3.1 (L^p – L^q off-diagonal estimate for a family of operators). Let $\{S_t\}_{t>0}$ be a family of operators. We say that $\{S_t\}_{t>0}$ satisfy the L^p – L^q off-diagonal estimate for some $p, q \in [1, \infty)$ with $p \leq q$ if there exist constants $C, c, \beta > 0$ such that for all closed sets $E, F \subset \mathbb{R}^n$, $t > 0$ and $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ supported in E , the following estimate holds:

$$\|S_t f\|_{L^q(F)} \leq C t^{\frac{1}{2m}(\frac{n}{q} - \frac{n}{p})} e^{-\left(\frac{d(E,F)}{ct^{1/2m}}\right)^\beta} \|f\|_{L^p(E)}, \quad (3.1)$$

where and in the sequel, $d(E, F)$ denotes the semi-distance induced on sets by the Euclidean distance. In particular, if (3.1) holds for $p = q$, then we say $\{S_t\}_{t>0}$ satisfy the L^p off-diagonal estimate.

Definition 3.2 (L^p – L^q estimate for a family of operators). Let $\{S_t\}_{t>0}$ be a family of operators. We say that $\{S_t\}_{t>0}$ satisfy an L^p – L^q estimate for some $p, q \in [1, \infty)$ with $p \leq q$ if

$$\|S_t f\|_{L^q} \leq C t^{\frac{1}{2m}(\frac{n}{q} - \frac{n}{p})} \|f\|_{L^p},$$

where $C > 0$, independent of t and $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Obviously, if $\{S_t\}_{t>0}$ satisfy an L^p – L^p estimate, then $\{S_t\}_{t>0}$ are bounded on L^p uniformly in t . In this case, we say S_t is bounded on L^p .

Lemma 3.1. *If two families of operators $\{S_s\}_{s>0}$ and $\{T_t\}_{t>0}$ satisfy L^2 off-diagonal estimates with different β_S and β_T in (3.1), respectively, then there exist $c, C > 0$ such that*

$$\|S_s T_t f\|_{L^2(F)} \leq C \max(e^{-(\frac{d(E,F)}{ct^{1/2m}})^{\beta_T}}, e^{-(\frac{d(E,F)}{cs^{1/2m}})^{\beta_S}}) \|f\|_{L^2(E)} \quad (3.2)$$

for any closed sets E, F in \mathbb{R}^n , $s, t > 0$ and $f \in L^2(\mathbb{R}^n)$ supported in E . In particular, if $\beta_S = \beta_T = \beta$ we have

$$\|S_s T_t f\|_{L^2(F)} \leq C e^{-(\frac{d(E,F)}{c \max(t^{1/2m}, s^{1/2m})})^\beta} \|f\|_{L^2(E)}. \quad (3.3)$$

Proof. This lemma is essentially due to [41]. For arbitrary closed sets E, F , let $\delta = d(E, F)$ and $G = \{x \in \mathbb{R}^n; d(x, E) < \frac{\delta}{3}\}$. Then by assumptions, for $s, t > 0$ and all $f \in L^2(\mathbb{R}^n)$ supported in E we have

$$\begin{aligned} \|S_s T_t f\|_{L^2(F)} &\leq \|S_s(\chi_G T_t f)\|_{L^2(F)} + \|S_s(\chi_{G^c} T_t f)\|_{L^2(F)} \\ &\leq C e^{-(\frac{d(G,F)}{cs^{1/2m}})^{\beta_S}} \|T_t f\|_{L^2(G)} + C \|T_t f\|_{L^2(G^c)} \\ &\leq C e^{-(\frac{d(G,F)}{cs^{1/2m}})^{\beta_S}} \|f\|_{L^2(E)} + C e^{-(\frac{d(G,F)}{ct^{1/2m}})^{\beta_T}} \|f\|_{L^2(E)} \\ &\leq C \max(e^{-(\frac{d(E,F)}{ct^{1/2m}})^{\beta_T}}, e^{-(\frac{d(E,F)}{cs^{1/2m}})^{\beta_S}}) \|f\|_{L^2(E)}. \end{aligned}$$

We hence get (3.2). \square

Lemma 3.2. *If $\{T_t\}_{t>0}$ satisfy L^p – L^q estimates (resp. off-diagonal estimates for β) and $\{S_t\}_{t>0}$ satisfy L^q – L^r estimates (resp. off-diagonal estimates for β), then $\{S_t T_t\}_{t>0}$ satisfy L^p – L^r estimates (resp. off-diagonal estimates for β).*

Proof. The proof of Lemma 3.2 is quite similar with the proof of Lemma 3.1, we omit the details here. \square

3.2. Some L^p – L^q off-diagonal estimates related to the semigroup e^{-tL}

In this subsection, we consider the L^p – L^q off-diagonal estimates of semigroup $\{e^{-tL}\}_{t \geq 0}$ generated by the following homogeneous higher order elliptic operator of order $2m \geq 4$ in divergence form:

$$Lf := (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha,\beta} \partial^\beta f),$$

where all complex coefficients $a_{\alpha,\beta} \in L^\infty(\mathbb{R}^n, \mathbb{C})$ and L is defined by the following sesquilinear form:

$$\mathcal{Q}(f, g) := \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha,\beta}(x) \partial^\beta f(x) \overline{\partial^\alpha g(x)} dx$$

which satisfies inequalities (1.5) and (1.6). Since L is a maximal accretive operator with $\mathcal{D}(L) \subseteq W^{m,2}(\mathbb{R}^n)$, where $\mathcal{D}(L)$ denotes the domain of L , the C_0 semigroup $\{e^{-tL}\}_{t \geq 0}$ generated by L on $L^2(\mathbb{R}^n)$ can be extended into an analytic semigroup $\{e^{-zL}\}_{z \in \sum_{\frac{\pi}{2}-\omega}}$ for some $\omega \in [0, \frac{\pi}{2})$ and enjoys a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. Here the set $\sum_{\mu} := \{z \in \mathbb{C}; |\arg z| < \mu\}$ for some $\mu > 0$.

For $t > 0$, it follows from Schwartz kernel theorem that the operator e^{-tL} has a distributional kernel $K_t(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$. As we know, it is very interesting to obtain further the pointwise estimates of $K_t(x, y)$ which are useful to study L^p -extension of semigroup e^{-tL} , L^p -spectra theory of L and other related problems. Indeed, when $n \leq 2m$, it was well known that the operator e^{-tL} has a kernel $K_t(x, y)$ satisfying the following pointwise estimate (see Davies [20], Barbatis and Davies [12] and Auscher et al. [8])

$$|K_t(x, y)| \leq Ct^{-\frac{n}{2m}} \exp\left(-c \frac{|x-y|}{t^{1/2m}}\right)^{\frac{2m}{2m-1}}, \quad (3.4)$$

which permits that the semigroup e^{-tL} defined on L^2 can be extended into a strongly continuous semigroup on $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$ and also bounded on $L^\infty(\mathbb{R}^n)$. However, when $n > 2m \geq 4$, the semigroup e^{-tL} on L^2 may be only extended to L^p for p belonging to finite interval around 2 unless all coefficients of L have some proper regularity, or uniform continuity (see e.g. Davies [22] and Auscher and Qafsaoui [9]). This certainly disproves the existence of such estimates as (3.4) as $n > 2m \geq 4$ under assumption of bounded measurability. Finally, note that if the semigroup e^{-tL} for $t \geq 0$ is a strongly continuous on $L^p(\mathbb{R}^n)$, then it is also analytic for $z \in \sum_{\frac{\pi}{2}-\omega_p}$ on L^p where $\omega_p \in [0, \frac{\pi}{2})$ depending on specific p (see [2, Proposition 3.15]).

We first give some important properties on the L^p – L^2 boundedness for the semigroup e^{-tL} , which is the version of Proposition 4.2 in [2] for the higher order operator L .

Theorem 3.1. *Let $p \in [1, 2)$ and L is defined by (1.4)–(1.6) with $2m < n$.*

- (i) *If e^{-tL} is bounded on L^p , then it satisfies the L^p – L^2 estimates.*
- (ii) *If e^{-tL} satisfies the L^p – L^2 estimate, then for all $p < q < 2$ it satisfies L^q – L^2 off-diagonal estimates.*
- (iii) *If e^{-tL} satisfies L^p – L^2 off-diagonal estimates, then it is bounded on L^p .*

Proof. Notice that the proofs of the conclusions (ii) and (iii) are essentially similar to one in [2], we hence give only the proof of (i) here.

To this end, we recall the Gagliardo–Nirenberg inequality (see e.g. Maz'ya [51, Section 2.3.12])

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla^m f\|_{L^2(\mathbb{R}^n)}^\alpha \|f\|_{L^p(\mathbb{R}^n)}^{1-\alpha}, \quad (3.5)$$

where $n > 2m$, $1 \leq p < 2$ and $\alpha = \delta_p / (1 + \delta_p)$ with $\delta_p = \frac{n}{m}(\frac{1}{p} - \frac{1}{2})$. Thus for $t > 0$ and $f \in L^2 \cap L^p$, by (3.5) we have

$$\|e^{-tL} f\|_{L^2(\mathbb{R}^n)}^2 \leq C \|\nabla^m e^{-tL} f\|_{L^2(\mathbb{R}^n)}^{2\alpha} \|e^{-tL} f\|_{L^p(\mathbb{R}^n)}^{2\beta}, \quad (3.6)$$

where $\alpha + \beta = 1$. By the strong Gårding inequality (1.6), we see that

$$\begin{aligned} \|\nabla^m e^{-tL} f\|_{L^2(\mathbb{R}^n)}^2 &\leq \lambda^{-1} \Re e \left(\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha,\beta}(x) \partial^\beta e^{-tL} f(x) \partial^\alpha \overline{e^{-tL} f(x)} dx \right) \\ &= -(2\lambda)^{-1} \frac{d}{dt} \|e^{-tL} f\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (3.7)$$

Now assume that $f \in L^2 \cap L^p$ with $\|f\|_{L^p} = 1$ and denote $\varphi(t) = \|e^{-tL} f\|_{L^2}^2$, then combining (3.6) and (3.7) with the idea of [2], we get (note that $\alpha = \delta_p/(1 + \delta_p)$)

$$\varphi(t) \leq C t^{\frac{\alpha}{\alpha-1}} = C t^{-\delta_p} = C t^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{2})}.$$

We therefore finish the proof of (i). \square

Remark 3.1. As shown in [2], Theorem 3.1 may be applied when $2 < p \leq \infty$ by duality: replacing L^p – L^2 estimates by L^2 – L^p estimates everywhere.

Now we denote by (p_L, \tilde{p}_L) the interior of the interval of L^p boundedness of the semigroup $\{e^{-tL}\}$, where

$$p_L := \inf \left\{ p \geq 1: \sup_{t>0} \|e^{-tL}\|_{L^p \rightarrow L^p} < \infty \right\} \quad (3.8)$$

and

$$\tilde{p}_L := \sup \left\{ p \leq \infty: \sup_{t>0} \|e^{-tL}\|_{L^p \rightarrow L^p} < \infty \right\}. \quad (3.9)$$

In [2, §7.2], Auscher pointed out (without proof) that $[2n/(n+2m), 2n/(n-2m)] \subset (p_L, \tilde{p}_L)$. As a consequence of Theorem 3.1, we give a proof of this fact for the completeness.

Corollary 3.1 (Auscher). *Suppose that L is defined by (1.4)–(1.6) with $2m < n$, then $[2n/(n+2m), 2n/(n-2m)] \subset (p_L, \tilde{p}_L)$.*

Proof. Let $f \in W^{m,q}(\mathbb{R}^n)$ and $g \in W^{m,q'}(\mathbb{R}^n)$ with $1/q + 1/q' = 1$, then

$$\begin{aligned} |\langle (L + \mathbf{I})f, g \rangle| &\leq \int_{\mathbb{R}^n} \left(\sum_{|\alpha|=|\beta|=m} |a_{\alpha\beta}(x) \partial^\alpha f(x) \overline{\partial^\beta g(x)}| + |f(x) \bar{g}(x)| \right) dx \\ &\leq \sum_{|\alpha|=|\beta|=m} \|a_{\alpha\beta}\|_\infty \int_{\mathbb{R}^n} |\partial^\alpha f(x) \overline{\partial^\beta g(x)}| dx + \int_{\mathbb{R}^n} |f(x) \bar{g}(x)| dx \\ &\leq C \|f\|_{W^{m,q}(\mathbb{R}^n)} \|g\|_{W^{m,q'}(\mathbb{R}^n)}. \end{aligned}$$

This means that $L + \mathbf{I}$ is bounded from $W^{m,q}$ to $W^{-m,q}$ for all $1 < q < \infty$. On the other hand, by Remark in [11, p. 11] we know that $(L + \mathbf{I})$ is invertible from $W^{m,2}$ to $W^{-m,2}$. Thus, applying a result of Sneiberg [58] (see also [11, Lemma 23]), there exists an $r_0 \in [1, 2)$, such that the operator $L + \mathbf{I}$ is bounded and invertible from $W^{m,q}$ to $W^{-m,q}$ with q satisfying $|1/2 - 1/q| < |1/2 - 1/r_0|$.

Denote $r_0^* = \frac{nr_0}{n+mr_0}$, then we claim that for $f \in L^2 \cap L^p$ with p satisfying

$$\begin{cases} 2 > p > r_0^*, & r_0^* \geq 1; \\ p = 1, & r_0^* < 1, \end{cases} \quad (3.10)$$

the following L^p – L^2 estimate holds:

$$\|e^{-tL} f\|_{L^2} + \|e^{-tL^*} f\|_{L^2} \leq C t^{-\frac{n}{2m}(1/p-1/2)} \|f\|_{L^p}. \quad (3.11)$$

If accepting (3.11), then by Theorem 3.1 (with Remark 3.1) and noting that $r_0^* = \frac{nr_0}{n+mr_0} < \frac{2n}{n+2m}$, we conclude immediately that, when $n > 2m$, the family of operators $\{e^{-tL}\}$ are L^p bounded for $p \in (\frac{2n}{n+2m} - \varepsilon, (\frac{2n}{n+2m} - \varepsilon)')$, where ε is some positive constant depending on n and the ellipticity constants. Thus we prove that $[2n/(n+2m), 2n/(n-2m)] \subset (p_L, \tilde{p}_L)$.

Hence, to complete the proof of this corollary, it remains to show (3.11). We first consider the case of $t = 1$. Write

$$e^{-L} f = e^{-L} (L + \mathbf{I})^k (L + \mathbf{I})^{-k} f,$$

where k will be chosen later. If we can show that

$$(L + \mathbf{I})^{-k} : L^2 \cap L^p \rightarrow L^2 \quad (3.12)$$

for the p defined in (3.10), then by the analyticity of the semigroup e^{-tL} , we can see that (3.11) holds for $t = 1$. To prove (3.12), by the Sobolev embedding theorem we have

$$L^{p_0} \hookrightarrow W^{-m, q_0} \xrightarrow{(L+\mathbf{I})^{-1}} W^{m, q_0} \hookrightarrow L^{q_0} \hookrightarrow W^{-m, q_1} \xrightarrow{(L+\mathbf{I})^{-1}} W^{m, q_1} \hookrightarrow \dots \hookrightarrow L^{q_{k-1}}$$

where $p_0 = p$ and $q_{k-1} \leq \frac{np_0}{n-kmp_0}$. So, if we choose $k \in \mathbb{N}$ such that $q_{k-1} = 2 \leq \frac{np_0}{n-kmp_0}$, then (3.12) holds and (3.11) follows for $t = 1$.

Now we show that (3.11) still is true for $t \neq 1$. Let U be the operator $Uf(x) = f(t^{-1/2m}x)$, denote $\tilde{L} = tU^{-1}LU$. Then $\{a_{\alpha\beta}(t^{-1/2m}x)\}$ are the coefficients of \tilde{L} , this means that (3.11) still holds for \tilde{L} with $t = 1$. Then we get that

$$\begin{aligned} \|e^{-tL} f\|_{L^2(\mathbb{R}^n)} &= \|Ue^{-\tilde{L}}U^{-1}f\|_{L^2(\mathbb{R}^n)} \leq C t^{\frac{n}{4m}} \|e^{-\tilde{L}}U^{-1}f\|_{L^2(\mathbb{R}^n)} \\ &\leq C t^{\frac{n}{4m} - \frac{n}{2mp}} \|f\|_{L^p(\mathbb{R}^n)}. \quad \square \end{aligned}$$

Now we give some off-diagonal estimates about the semigroup e^{-tL} as follows:

Theorem 3.2. For the semigroup $\{e^{-tL}\}_{t \geq 0}$, the following statements hold:

- (i) The families $\{tLe^{-tL}\}_{t>0}$ and $\{t^{\frac{k}{2m}}\nabla^k e^{-tL}\}_{t>0}$ ($k = 0, 1, 2, \dots, m$) satisfy L^2 off-diagonal estimates (3.1) with $\beta = \frac{2m}{2m-1}$.
- (ii) The family $\{(\mathbf{I} + tL)^{-1}\}_{t>0}$ satisfies L^2 off-diagonal estimates (3.1) with $\beta = 1$.

- (iii) Given $t > 0$, the families $\{sLe^{-(s+t)L}\}_{s \geq t}$ and $\{\frac{s}{t}(e^{-sL} - e^{-(s+t)L})\}_{s \geq t}$ satisfy L^2 off-diagonal estimates (3.1) with $\beta = \frac{2m}{2m-1}$.
- (iv) For $p_L < p \leq 2 \leq q < \tilde{p}_L$, the families $\{e^{-tL}\}_{t>0}$ and $\{tLe^{-tL}\}_{t>0}$ satisfy L^p – L^q off-diagonal estimates (3.1) with $\beta = \frac{2m}{2m-1}$.

Proof. The conclusion (i) about the family of operators $\{t^{\frac{k}{2m}} \nabla^k e^{-tL}\}_{t>0}$ ($k = 0, 1, 2, \dots, m$) can be found in [2, p. 66]. On the other hand, since e^{-tL} is an analytic semigroup in a sector containing the positive real axis on L^2 , we can conclude that the family of operators $\{tLe^{-tL}\}_{t>0}$ satisfy also the L^2 off diagonal estimate (see [2, §3.3]).

(ii) For any closed sets $E, F \subset \mathbb{R}^n$ with $d(E, F) > 0$ and $f \in L^2(\mathbb{R}^n)$ with its support set in E , applying the L^2 off-diagonal estimate for $\{e^{-tL}\}_{t>0}$ above and combining with the formula

$$(L + \lambda)^{-1}g = \int_0^\infty e^{-\lambda t} e^{-tL} g dt \quad \text{for } g \in L^2(\mathbb{R}^n),$$

we have

$$\begin{aligned} \|(\mathbf{I} + tL)^{-1}f\|_{L^2(F)} &= \|t^{-1}(t^{-1} + L)^{-1}f\|_{L^2(F)} \\ &\leq t^{-1} \int_0^\infty e^{-st^{-1}} \|e^{-sL}f\|_{L^2(F)} ds \\ &\leq C \int_0^\infty e^{-r} e^{-\left(\frac{d(E,F)}{c(rt)^{1/2m}}\right)^{\frac{2m}{2m-1}}} dr \|f\|_{L^2(E)} \\ &\leq C \left(\int_0^{t_0} e^{-r} e^{-\left(\frac{d(E,F)}{c(rt)^{1/2m}}\right)^{\frac{2m}{2m-1}}} dr + \int_{t_0}^\infty e^{-r} e^{-\left(\frac{d(E,F)}{c(rt)^{1/2m}}\right)^{\frac{2m}{2m-1}}} dr \right) \|f\|_{L^2(E)} \\ &\leq C \left(e^{-c't_0} \int_0^\infty e^{-r} dr + e^{-\frac{1}{2}t_0} \int_0^\infty e^{-\frac{1}{2}r} e^{-\left(\frac{d(E,F)}{c(rt)^{1/2m}}\right)^{\frac{2m}{2m-1}}} dr \right) \|f\|_{L^2(E)} \\ &\leq Ce^{-c't_0} \|f\|_{L^2(E)}, \end{aligned}$$

where $t_0 = t^{-\frac{1}{2m}} d(E, F)$.

We now prove the conclusion (iii). Let E and F be arbitrary closed sets in \mathbb{R}^n and $f \in L^2(\mathbb{R}^n)$ supported in E . For $s \geq t > 0$, we have

$$\begin{aligned} \|(sLe^{-(t+s)L})f\|_{L^2(F)} &= \frac{s}{s+t} \|(s+t)Le^{-(t+s)L}f\|_{L^2(F)} \\ &\leq Ce^{-\left(\frac{d(E,F)}{c(s+t)^{1/2m}}\right)^{\frac{2m}{2m-1}}} \frac{s}{s+t} \|f\|_{L^2(E)} \\ &\leq Ce^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right)^{\frac{2m}{2m-1}}} \|f\|_{L^2(E)} \end{aligned}$$

and

$$\begin{aligned}
 \|(e^{-sL} - e^{-(t+s)L})f\|_{L^2(F)} &= \left\| \int_0^t \partial_r e^{-(r+s)L} f \, dr \right\|_{L^2(F)} \\
 &\leq C \int_0^t \|(s+r)L e^{-(r+s)L} f\|_{L^2(F)} \frac{dr}{s+r} \\
 &\leq C \int_0^t e^{-\left(\frac{d(E,F)}{c(s+r)^{1/2m}}\right) \frac{2m}{2m-1}} \frac{dr}{s+r} \|f\|_{L^2(E)} \\
 &\leq C e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right) \frac{2m}{2m-1}} \int_0^t \frac{dr}{s} \|f\|_{L^2(E)} \\
 &\leq C \frac{t}{s} e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right) \frac{2m}{2m-1}} \|f\|_{L^2(E)},
 \end{aligned}$$

which implies

$$\left\| \frac{s}{t} (e^{-sL} - e^{-(t+s)L}) f \right\|_{L^2(F)} \leq C e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right) \frac{2m}{2m-1}} \|f\|_{L^2(E)}.$$

Finally, we show the conclusion (iv). By Theorem 3.1 and Remark 3.1, we see that e^{-tL} satisfies L^p – L^2 estimates for $p_L < p < 2$ and L^2 – L^p estimates for $2 < p < \tilde{p}_L$. Using the Riesz–Thorin interpolation and the L^2 off-diagonal estimate of e^{-tL} , we get that the family of operators $\{e^{-tL}\}_{t>0}$ satisfies L^p – L^q off-diagonal estimates for $p_L < p \leq 2 \leq q < \tilde{p}_L$ with $\beta = \frac{2m}{2m-1}$ in (3.1).

As for the family of the operators $\{tLe^{-tL}\}_{t>0}$, we may write $tLe^{-tL} = 2(\frac{t}{2}Le^{-\frac{t}{2}L})e^{-\frac{t}{2}L}$. Thus by Lemma 3.2, the family of operators $\{tLe^{-tL}\}_{t>0}$ satisfies the L^p – L^q off-diagonal estimates. \square

Remark 3.2. (i) When $m = 1$, L is a second order elliptic operator in divergence form. The corresponding results of Theorem 3.2 can be found in [1,2,4,41,42] etc.

(ii) The conclusion (iv) in Theorem 3.2 is also valid for all p, q satisfying $p_L < p < q < \tilde{p}_L$. In fact, it can be obtained by applying the Riesz–Thorin interpolation theorem between

$$\|e^{-tL}f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

and the conclusion of (iv). However, in this paper, we just need to use the results in these cases where $p_L < p \leq 2 \leq q < \tilde{p}_L$.

3.3. L^p – L^q estimates for vertical square function

We now define the vertical square function $g_h^{L,k}$ associated with the semigroup e^{-tL} . For $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$, $g_h^{L,k}$ is defined by

$$g_h^{L,k} f(x) := \left(\int_0^\infty |(tL)^k e^{-tL} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

When $k = 1$, we denote $g_h^{L,1}$ by g_h^L . We have the following useful L^p – L^q off-diagonal estimates:

Lemma 3.3. *Let $p_L < p < \tilde{p}_L$ and $k \in \mathbb{N}$. Then there exists $M \in \mathbb{N}$ and $M > \frac{n}{4m}(2m-1)$ such that for all closed sets $E, F \subset \mathbb{R}^n$ with $d(E, F) > 0$ the following results hold:*

(i) *If $p_L < p \leq 2$ and $f \in L^p(\mathbb{R}^n)$ supported in E , then*

$$\|g_h^{L,k}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} \leq C t^{\frac{1}{2m}(\frac{n}{2} - \frac{n}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}, \quad (3.13)$$

$$\|g_h^{L,k}(tLe^{-tL})^M f\|_{L^2(F)} \leq C t^{\frac{1}{2m}(\frac{n}{2} - \frac{n}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}. \quad (3.14)$$

(ii) *If $2 \leq p < \tilde{p}_L$ and $f \in L^2(\mathbb{R}^n)$ supported in E , then*

$$\|g_h^{L,k}(\mathbf{I} - e^{-tL})^M f\|_{L^p(F)} \leq C t^{\frac{1}{2m}(\frac{n}{p} - \frac{n}{2})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}, \quad (3.15)$$

$$\|g_h^{L,k}(tLe^{-tL})^M f\|_{L^p(F)} \leq C t^{\frac{1}{2m}(\frac{n}{p} - \frac{n}{2})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}. \quad (3.16)$$

Proof. The conclusion of Lemma 3.3 for the case $m = 1$ may be found in the remark in [42, p. 56]. Since the proof of the conclusion (ii) is similar, so we only give the proof of the conclusion (i) for $m > 1$. By the L^2 theory of quadratic estimates for operators having a bounded holomorphic functional calculus on L^2 , we know that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_{L^2} \leq \|g_h^{L,k} f\|_{L^2} \leq c_2 \|f\|_{L^2}. \quad (3.17)$$

(See [11] or [52], for example.)

We first prove that (3.13) and (3.14) hold for $k = 1$. Let E and F be closed sets with $d(E, F) > 0$ and $f \in L^p(\mathbb{R}^n)$ with $\text{supp } f \subset E$. Write

$$\|g_h^L(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} = \left\| \left(\int_0^\infty |sLe^{-sL}(\mathbf{I} - e^{-tL})^M f|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^2(F)}$$

$$\begin{aligned}
&\leq C \left(\int_0^\infty \|sLe^{-s(M+1)L}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t \|sLe^{-s(M+1)L}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\quad + C \left(\int_t^\infty \|sLe^{-s(M+1)L}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&:= I_1 + I_2.
\end{aligned}$$

For I_1 , using the binomial formula for $(\mathbf{I} - e^{-tL})^M$ and (iv) of Theorem 3.2

$$\begin{aligned}
I_1 &\leq C \left(\int_0^t \|sLe^{-s(M+1)L} f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\quad + C \sup_{1 \leq k \leq M} \left(\int_0^t \|sLe^{-s(M+1)L} e^{-ktL} f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t s^{\frac{n}{m}(\frac{1}{2} - \frac{1}{p})} e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right)^{\frac{2m}{2m-1}} \frac{ds}{s}} \right)^{\frac{1}{2}} \|f\|_{L^p(E)} \\
&\quad + C \sup_{1 \leq k \leq M} \left(\int_0^t \left\| e^{-s(M+1)L} \left(\frac{k}{2} tLe^{-\frac{k}{2}tL} \right) e^{-\frac{k}{2}tL} f \right\|_{L^2(F)}^2 \frac{s ds}{t^2} \right)^{\frac{1}{2}}. \quad (3.18)
\end{aligned}$$

For the second term of (3.18), by Lemma 3.1 we have

$$\left\| e^{-s(M+1)L} \left(\frac{k}{2} tLe^{-\frac{k}{2}tL} \right) f \right\|_{L^2(F)} \leq C e^{-\left(\frac{d(E,F)}{ct^{1/2m}}\right)^{\frac{2m}{2m-1}}} \|f\|_{L^2(E)}.$$

Hence it follows that from (iv) of Theorem 3.2 and Lemma 3.2 that $e^{-s(M+1)L} \left(\frac{k}{2} tLe^{-\frac{k}{2}tL} \right) e^{-\frac{k}{2}tL}$ satisfies the L^p – L^2 off-diagonal estimates, and the second term of (3.18) is bounded by

$$\begin{aligned}
&t^{\frac{1}{2m}(\frac{n}{2} - \frac{n}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \left(\int_0^t \frac{s ds}{t^2} \right)^{\frac{1}{2}} \|f\|_{L^p(E)} \\
&\leq C t^{\frac{1}{2m}(\frac{n}{2} - \frac{n}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}.
\end{aligned}$$

Thus

$$I_1 \leq C t^{\frac{1}{2m}(\frac{n}{2} - \frac{n}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}. \quad (3.19)$$

For I_2 , use (iii) of Theorem 3.2 for the operator $e^{-sL} - e^{-(t+s)L}$ and Lemma 3.2, we get

$$\begin{aligned} I_2 &\leq C \left(\int_t^\infty \|sL e^{-sL} (e^{-sL} - e^{-(t+s)L})^M f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\leq C \left(\int_t^\infty \left(\frac{t}{s} \right)^{2M} \left\| sL e^{-sL} \left(\frac{s}{t} (e^{-sL} - e^{-(t+s)L}) \right)^M f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\leq C \left(\int_t^\infty \left(\frac{t}{s} \right)^{2M} s^{\frac{n}{m}(\frac{1}{2} - \frac{1}{p})} e^{-\left(\frac{d(E, F)}{cs^{1/2m}}\right)^{\frac{2m}{2m-1}}} \frac{ds}{s} \right)^{\frac{1}{2}} \|f\|_{L^p(E)} \\ &\leq C t^M \left(\int_t^\infty \left(\frac{s}{d(E, F)^{2m}} \right)^{2M/(2m-1)} s^{-2M + \frac{n}{m}(\frac{1}{2} - \frac{1}{p})} \frac{ds}{s} \right)^{1/2} \|f\|_{L^p(E)} \\ &\leq C t^{\frac{n}{2m}(\frac{1}{2} - \frac{1}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}. \end{aligned} \quad (3.20)$$

Thus, (3.13) holds for $k = 1$. The proof for (3.14) is essentially similar. More precisely, we only need to estimate the integrals I_1 and I_2 with $(\mathbf{I} - e^{-tL})^M$ replaced by $(tLe^{-tL})^M$. Here we omit these repeated arguments. Hence the conclusion (i) holds for the case $k = 1$.

Below we consider the case where $k > 1$. Let $s := rk(M+1)$, then we have

$$\begin{aligned} \|g_h^{L,k}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} &= \left\| \left(\int_0^\infty |(sL)^k e^{-sL} (\mathbf{I} - e^{-tL})^M f|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^2(F)} \\ &\leq C_{M,k} \left(\int_0^\infty \|(rL)^k e^{-kr(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 \frac{dr}{r} \right)^{\frac{1}{2}} \\ &\leq C_{M,k} (J_1 + J_2), \end{aligned}$$

where

$$J_1 := \left(\int_0^t \|(rL e^{-rL} \cdot e^{-rML})^{k-1} (rL e^{-r(M+1)L} (\mathbf{I} - e^{-tL})^M f)\|_{L^2(F)}^2 \frac{dr}{r} \right)^{\frac{1}{2}},$$

and

$$J_2 := \left(\int_t^\infty \left\| (rLe^{-rL} \cdot e^{-rML})^{k-1} (rLe^{-r(M+1)L} (\mathbf{I} - e^{-tL})^M f) \right\|_{L^2(F)}^2 \frac{dr}{r} \right)^{\frac{1}{2}}.$$

Using the conclusion (i) of Theorem 3.2 for the operators rLe^{-rL} and e^{-rML} on $L^2(F)$, it is easy to see that $J_j \leq C_{M,k} I_j$ for $j = 1, 2$. Hence, (3.13) holds for the case $k > 1$. Similarly, we can show that (3.14) holds also for $k > 1$. We therefore finish the proof of Lemma 3.3. \square

3.4. Two general theorems for L^p theory of Calderón–Zygmund type operators

For a ball $B \subset \mathbb{R}^n$ and $\lambda > 0$, we denote by λB the ball with same center and radius λ times that of B . We set

$$S_1(B) = 4B, \quad S_j(B) = 2^{j+1}B \setminus 2^j B \quad \text{for } j \geq 2.$$

Denote by \mathcal{M} the Hardy–Littlewood maximal operator

$$\mathcal{M}(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy$$

where B ranges over all open balls (or cubes) containing x . The following two theorems are very useful in dealing with the L^p boundedness of operators.

Theorem A (Blunck–Kunzmann). *Let $p_0 \in [1, 2)$. Suppose that T is a sublinear operator of strong type $(2, 2)$ and $\{A_r\}_{r>0}$ is a family of linear operators acting on $L^2(\mathbb{R}^n)$. If for $j \geq 2$*

$$\left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |T(I - A_{r(B)})f(x)|^2 dx \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}}, \quad (3.21)$$

and for $j \geq 1$

$$\left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |A_{r(B)}f(x)|^2 dx \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}}, \quad (3.22)$$

for all ball B with $r(B)$ the radius of B and all f supported in B . If $\sum_j g(j)2^{jn} < \infty$, then T is of weak type (p_0, p_0) , with the bound depending only on the strong type $(2, 2)$ bound of T , p_0 and the sum $\sum_j g(j)2^{jn}$. Hence, by interpolation T also is bounded on $L^p(\mathbb{R}^n)$ for $p_0 < p < 2$.

Theorem B (Auscher–Coulhon–Duong–Hofmann). *Let $p_0 \in [2, \infty)$. Suppose that T is a sublinear operator acting on $L^2(\mathbb{R}^n)$ and $\{A_r\}_{r>0}$ is a family of linear operators acting on $L^2(\mathbb{R}^n)$. Also assume that*

$$\left(\frac{1}{|B|} \int_B |T(I - A_{r(B)})f(x)|^2 dx \right)^{\frac{1}{2}} \leq C(\mathcal{M}(|f|^2))^{\frac{1}{2}}(y) \quad (3.23)$$

and

$$\left(\frac{1}{|B|} \int_B |T A_{r(B)}f(x)|^{p_0} dx \right)^{\frac{1}{p_0}} \leq C(\mathcal{M}(|Tf|^2))^{\frac{1}{2}}(y) \quad (3.24)$$

for all $f \in L^2$, all ball B and all $y \in B$ where $r(B)$ is the radius of B . If $2 < p < p_0$ and $Tf \in L^p$ as $f \in L^p$, then T is of strong type (p, p) . That is, for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)},$$

where c depends only on n , p , and p_0 and C .

Theorems A and B are essentially due to [13, Theorem 1.1, p. 920] and [3, Theorem 2.1, p. 923], respectively. See [2] for the proofs and nice comments on Theorems A and B.

3.5. L^p boundedness of the square functions $g_h^{L,k}$ and $S_h^{L,k}$

In this subsection, we will give the L^p boundedness of the vertical square function $g_h^{L,k}$ and area integral $S_h^{L,k}$. We first consider $g_h^{L,k}$.

Theorem 3.3. Let $p_L < p < \tilde{p}_L$, where p_L and \tilde{p}_L are defined in (3.8) and (3.9), respectively. Then the vertical square function $g_h^{L,k}$ is bounded on $L^p(\mathbb{R}^n)$ for $k \in \mathbb{N}$.

Proof. By (3.17) we know that $g_h^{L,k}$ is an operator of type $(2, 2)$.

(i) Case where $p_L < p < 2$. Let B be a ball and $r = r(B)$ its radius. Choose $A_r = \mathbf{I} - (\mathbf{I} - e^{-r^{2m}L})^M$ for $M \in \mathbb{N}$ and $M > \frac{n}{4m}(2m - 1)$. Thus, by Theorem A, we only need to show that (3.21) and (3.22) hold for $g_h^{L,k}$ and p_0 , where p_0 satisfies $p_L < p_0 < p < 2$.

We first verify (3.22). Let $j \geq 1$ and f supported in B . Notice that $A_r = \sum_{\ell=1}^M C_{M,\ell} e^{-\ell r^{2m}L}$. For $\ell = 1, 2, \dots, M$,

$$\begin{aligned} & \left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |e^{-\ell r^{2m}L} f(x)|^2 dx \right)^{\frac{1}{2}} \\ & \leq C 2^{-\frac{jn}{2}} |B|^{-\frac{1}{2}} \|e^{-\ell r^{2m}L} f\|_{L^2(S_j(B))} \\ & \leq C 2^{-\frac{jn}{2}} |B|^{-\frac{1}{2}} r^{-n(\frac{1}{p_0} - \frac{1}{2})} e^{-\left(\frac{d(B, S_j(B))^{2m}}{r^{2m}}\right)^{\frac{1}{2m-1}}} \|f\|_{L^{p_0}(B)} \\ & \leq C 2^{-\frac{jn}{2}} e^{-c 2^{\frac{2jm}{2m-1}}} \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}}, \end{aligned} \quad (3.25)$$

where we use Theorem 3.2 in the second inequality. Thus (3.25) shows that

$$\left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |A_{r(B)} f(x)|^2 dx \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}},$$

with $\sum_j g(j)2^{jn} < \infty$. Next we establish (3.21).

$$\begin{aligned} & \left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |g_h^{L,k}(\mathbf{I} - A_{r(B)}) f(x)|^2 dx \right)^{\frac{1}{2}} \\ & \leq C 2^{-\frac{jn}{2}} |B|^{-\frac{1}{2}} \|g_h^{L,k}(\mathbf{I} - e^{-r^{2m}L})^M f\|_{L^2(S_j(B))} \\ & \leq C 2^{-\frac{jn}{2}} |B|^{-\frac{1}{2}} r^{-n(\frac{1}{p_0} - \frac{1}{2})} \left(\frac{r^{2m}}{d(B, S_j(B))^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^{p_0}(B)} \\ & \leq C 2^{-j(\frac{n}{2} + \frac{2mM}{2m-1})} \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}}, \end{aligned}$$

where we use (3.13). Then when $M \in \mathbb{N}$, $M > \frac{n}{4m}(2m-1)$, we have $\sum_j g(j)2^{jn} < \infty$ by taking $g(j) = C 2^{-j(\frac{n}{2} + \frac{2mM}{2m-1})}$. Hence, $g_h^{L,k}$ is bounded on $L^p(\mathbb{R}^n)$ by Theorem A.

(ii) *Case where $2 < p < \tilde{p}_L$.* As before, let $A_r = \mathbf{I} - (\mathbf{I} - e^{-r^{2m}L})^M$, where $r = r(B)$ is the radius of the ball B . Below we show (3.23) and (3.24) hold for $g_h^{L,k}$ and p_0 , where p_0 is taken to satisfy $2 < p < p_0 < \tilde{p}_L$. Here we denote $\tilde{S}_j(B)$ by

$$\tilde{S}_j(B) = 2^{j+1}B \setminus 2^jB \quad \text{for } j \geq 0, \quad \tilde{S}_{-1}(B) = B.$$

For $j \geq 2$, the definition of $\tilde{S}_j(B)$ given here coincides with $S_j(B)$ in Section 3.4. We check (3.24) firstly. Notice that $A_r = \sum_{\ell=1}^M C_{M,\ell} e^{-\ell r^{2m}L}$ and for $\ell = 1, 2, \dots, M$ and any $y \in B$, by Minkowski's inequality, we have

$$\begin{aligned} & \left(\frac{1}{|B|} \int_B |g_h^{L,k} e^{-\ell r^{2m}L} f(x)|^{p_0} dx \right)^{\frac{2}{p_0}} \\ & = \left(\frac{1}{|B|} \int_B \left| \int_0^\infty |(t^{2m}L)^k e^{-t^{2m}L} e^{-\ell r^{2m}L} f(x)|^2 \frac{dt}{t} \right|^{\frac{p_0}{2}} dx \right)^{\frac{2}{p_0}} \\ & \leq |B|^{-\frac{2}{p_0}} \int_0^\infty \|e^{-\ell r^{2m}L} ((t^{2m}L)^k e^{-t^{2m}L} f)\|_{L^{p_0}(B)}^2 \frac{dt}{t}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & |B|^{-\frac{1}{p_0}} \|e^{-\ell r^{2m} L} ((t^{2m} L)^k e^{-t^{2m} L} f)\|_{L^{p_0}(B)} \\
 & \leq \sum_{j \geq -1} \left(\frac{1}{|B|} \int_B |e^{-\ell r^{2m} L} (\chi_{\tilde{S}_j(B)} (t^{2m} L)^k e^{-t^{2m} L} f)(x)|^{p_0} dx \right)^{\frac{1}{p_0}} \\
 & \leq \sum_{j \geq -1} 2^{\frac{n(j+1)}{2}} e^{-c 2^{\frac{2mj}{2m-1}}} \left(\frac{1}{|2^{j+1} B|} \int_{\tilde{S}_j(B)} |(t^{2m} L)^k e^{-t^{2m} L} f(x)|^2 dx \right)^{\frac{1}{2}},
 \end{aligned}$$

where we use Theorem 3.2 in the second inequality. Hence, we obtain

$$\begin{aligned}
 & \left(\frac{1}{|B|} \int_B |g_h^{L,k} e^{-\ell r^{2m} L} f(x)|^{p_0} dx \right)^{\frac{2}{p_0}} \\
 & \leq C \sum_{j \geq -1} 2^{\frac{n(j+1)}{2}} e^{-c 2^{\frac{2mj}{2m-1}}} \frac{1}{|2^{j+1} B|} \int_0^\infty \int_{\tilde{S}_j(B)} |(t^{2m} L)^k e^{-t^{2m} L} f(x)|^2 dx \frac{dt}{t} \\
 & \leq C \sum_{j \geq -1} 2^{\frac{n(j+1)}{2}} e^{-c 2^{\frac{2mj}{2m-1}}} \frac{1}{|2^{j+1} B|} \|g_h^{L,k} f\|_{L^2(2^{j+1} B)}^2 \\
 & \leq C \mathcal{M}(|g_h^{L,k} f|^2)(y),
 \end{aligned}$$

which implies the inequality (3.24). As for (3.23), let $f \in L^2(\mathbb{R}^n)$ and $y \in B$. One has

$$\begin{aligned}
 & \left(\frac{1}{|B|} \int_B |g_h^{L,k} (\mathbf{I} - A_{r(B)}) f(x)|^2 dx \right)^{\frac{1}{2}} \\
 & = |B|^{-\frac{1}{2}} \|g_h^{L,k} (\mathbf{I} - e^{-r^{2m} L})^M f\|_{L^2(B)} \\
 & \leq \sum_{j \geq 1} |B|^{-\frac{1}{2}} \|g_h^{L,k} (\mathbf{I} - e^{-r^{2m} L})^M (\chi_{S_j(B)} f)\|_{L^2(B)} \\
 & \leq C \sum_{j \geq 1} |B|^{-\frac{1}{2}} 2^{-j \frac{2mM}{2m-1}} \|f\|_{L^2(S_j(B))} \\
 & \leq C \sum_{j \geq 1} 2^{-j (\frac{2mM}{2m-1} - \frac{n}{2})} (\mathcal{M}(|f|^2))^{\frac{1}{2}}(y),
 \end{aligned}$$

where we use (3.13) with $p = 2$ in the third inequality above. Since $M > \frac{n}{4m}(2m-1)$, we get

$$\left(\frac{1}{|B|} \int_B |g_h^{L,k} (\mathbf{I} - A_{r(B)}) f(x)|^2 dx \right)^{\frac{1}{2}} \leq C (\mathcal{M}(|f|^2))^{\frac{1}{2}}(y). \quad (3.26)$$

Thus $g_h^{L,k}$ is bounded on $L^p(\mathbb{R}^n)$ by Theorem B. We therefore finish the proof of Theorem 3.3. \square

Before stating the definition and the L^p boundedness of the area integrals $S_h^{L,k}$, let us recall some definitions and results related to tent spaces, which were given by Coifman, Meyer and Stein in [15].

For $\alpha > 0$ and $x \in \mathbb{R}^n$, let Γ^α be the cone of aperture α and vertex at x , i.e. for

$$\Gamma^\alpha(x) := \{(y, t) \in \mathbb{R}^n \times (0, \infty): |x - y| < \alpha t\}.$$

For any closed set F in \mathbb{R}^n , denote by $\mathcal{R}^\alpha(F) := \bigcup_{x \in F} \Gamma^\alpha(x)$. We also often write $\Gamma(x)$ (resp. $\mathcal{R}(F)$) in place of $\Gamma^1(x)$ (resp. $\mathcal{R}^1(F)$) for simplicity.

Let F be a closed set whose complement O has finite measure and $0 < \gamma < 1$. We say that a point $\bar{x} \in \mathbb{R}^n$ has a global γ -density with respect to F , if $|F \cap B(\bar{x})| \geq \gamma$ for any balls $B(\bar{x})$ centered at \bar{x} . Denote by F^\circledast all the points of having global γ -density with respect to F . Then F^\circledast is closed, and $F^\circledast \subset F$. Let $O^\circledast = (F^\circledast)^c$, then $O \subset O^\circledast$, but $|O^\circledast| \leq c_\gamma |O|$. (See [15, p. 310] for the above facts.) The following lemma can be found in [15]. (See [15, Lemmas 1 and 2].)

Lemma 3.4 (Coifman–Meyer–Stein). *Suppose that $\alpha > 0$ is given. Then there exists a $0 < \gamma < 1$ so that whenever F is a closed set whose complement has finite measure, and Φ is nonnegative, we have*

$$\int_{\mathcal{R}^\alpha(F^\circledast)} \Phi(y, t) t^n dy dt \leq c_{\alpha, \gamma} \int_F \left\{ \int_{\Gamma^\alpha(x)} \Phi(y, t) dy dt \right\} dx, \quad (3.27)$$

where F^\circledast denotes the set of point of having global γ -density with respect to F .

Conversely, for every closed set F in \mathbb{R}^n and every nonnegative function Φ ,

$$\int_F \left\{ \int_{\Gamma^\alpha(x)} \Phi(y, t) dy dt \right\} dx \leq c_\alpha \int_{\mathcal{R}^\alpha(F)} \Phi(y, t) t^n dy dt. \quad (3.28)$$

Now we give the definition of the area integral $S_h^{L,k}$ associated to the semigroup e^{-tL} , where L is the homogeneous elliptic operator of $2m$ order in divergence form defined by (1.4)–(1.6). For $f \in L^2(\mathbb{R}^n)$

$$S_h^{L,k} f(x) := \left(\iint_{\Gamma(x)} |(t^{2m} L)^k e^{-t^{2m} L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n, k \in \mathbb{N}. \quad (3.29)$$

We will denote $S_h^{L,1}$ by S_h^L for simplicity.

Theorem 3.4. *Let $p_L < p < \tilde{p}_L$. Then for $k \in \mathbb{N}$, the area integral $S_h^{L,k}$ is bounded on $L^p(\mathbb{R}^n)$.*

Proof. The proof closely follows an argument analogous to the one in the proof of Theorem 3.3. First, for $k \geq 1$ and $f \in L^2(\mathbb{R}^n)$, using (3.28) and the L^2 boundedness of $g_h^{L,k}$ (see Theorem 3.3), we have

$$\|S_h^{L,k} f\|_{L^2} \leq C \left(\iint_{\mathbb{R}_+^{n+1}} |(t^{2m} L)^k e^{-t^{2m} L} f(y)|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} \leq C \|g_h^{L,k} f\|_{L^2} \leq C \|f\|_{L^2}.$$

(i) *Case where $p_L < p < 2$.* To apply Theorem A, we denote $A_r = \mathbf{I} - (\mathbf{I} - e^{-r^{2m} L})^M$, where $M > \frac{n}{4m}(2m-1)$ and $r = r(B)$ is the radius of ball B . Then (3.25) shows that, to prove that $S_h^{L,k}$ is bounded on $L^p(\mathbb{R}^n)$ it suffices to show that (3.21) holds for $S_h^{L,k}$ and p_0 , where p_0 is taken to satisfy $p_L < p_0 < p < 2$. Let $\tilde{S}_j(B)$ ($j \geq -1$) be defined as in the proof of Theorem 3.3. For $j \geq 2$ and f supported in B , by (3.28) we obtain

$$\begin{aligned} & \int_{S_j(B)} |S_h^{L,k}(\mathbf{I} - A_{r(B)}) f(x)|^2 dx \\ &= \int_{S_j(B)} \iint_{\Gamma(x)} |(t^{2m} L)^k e^{-t^{2m} L} (\mathbf{I} - e^{-r^{2m} L})^M f(y)|^2 \frac{dy dt}{t^{n+1}} dx \\ &\leq C \iint_{\mathcal{R}(S_j(B))} |(t^{2m} L)^k e^{-t^{2m} L} (\mathbf{I} - e^{-r^{2m} L})^M f(x)|^2 \frac{dx dt}{t} \\ &\leq C \int_{\mathbb{R}^n \setminus 2^{j-1} B} \int_0^\infty |(t^{2m} L)^k e^{-t^{2m} L} (\mathbf{I} - e^{-r^{2m} L})^M f(x)|^2 \frac{dt}{t} dx \\ &\quad + C \sum_{\ell=-1}^{j-2} \int_{\tilde{S}_\ell(B)} \int_{(2^{j-2^{\ell+1}})r}^\infty |(t^{2m} L)^k e^{-t^{2m} L} (\mathbf{I} - e^{-r^{2m} L})^M f(x)|^2 \frac{dt}{t} dx \\ &:= I + \sum_{\ell=-1}^{j-2} I_\ell. \end{aligned}$$

For I , by (3.13) we have

$$\begin{aligned} I &= C \|g_h^{L,k}(\mathbf{I} - e^{-r^{2m} L})^M f\|_{L^2(\mathbb{R}^n \setminus 2^{j-1} B)}^2 \\ &\leq C r^{-2n(\frac{1}{p_0} - \frac{1}{2})} \left(\frac{r^{2m}}{d(B, \mathbb{R}^n \setminus 2^{j-1} B)^{2m}} \right)^{\frac{2M}{2m-1}} \|f\|_{L^{p_0}(B)}^2 \\ &\leq C r^{-2n(\frac{1}{p_0} - \frac{1}{2})} 2^{-\frac{4mjM}{2m-1}} \|f\|_{L^{p_0}(B)}^2. \end{aligned} \tag{3.30}$$

Now turn to I_ℓ for $\ell = -1, 0, 1, 2, \dots, j-2$. We make change of variables $\alpha := \frac{t^{2m}}{k(M+1)}$, then

$$\begin{aligned} I_\ell &\leq C_{M,k} \int_{c(2^j r)^{2m}}^{\infty} \|(\alpha L)^k e^{-\alpha k(M+1)L} (\mathbf{I} - e^{-r^{2m}L})^M(f)\|_{L^2(\tilde{S}_\ell(B))}^2 \frac{d\alpha}{\alpha} \\ &\leq C_{M,k} \int_{c(2^j r)^{2m}}^{\infty} \left(\frac{r^{2m}}{\alpha}\right)^{2M} \left\| \left(\frac{\alpha k}{r^{2m}}(e^{-\alpha kL} - e^{-(r^{2m}+\alpha k)L})\right)^M (\alpha L e^{-\alpha L})^k(f) \right\|_{L^2(\tilde{S}_\ell(B))}^2 \frac{d\alpha}{\alpha}. \end{aligned}$$

For $\ell \geq 1$, apply the L^2 off-diagonal estimates for $\frac{\alpha k}{r^{2m}}(e^{-\alpha kL} - e^{-(r^{2m}+\alpha k)L})$ (see (iii) in Theorem 3.2) and L^{p_0} – L^2 off-diagonal estimates for $\alpha L e^{-\alpha L}$ (see (iv) in Theorem 3.2), then by Lemma 3.2 we get

$$\begin{aligned} I_\ell &\leq C_{M,k} \int_{c(2^j r)^{2m}}^{\infty} \left(\frac{r^{2m}}{\alpha}\right)^{2M} \alpha^{\frac{1}{m}(\frac{n}{2} - \frac{n}{p_0})} e^{-\left(\frac{d(\tilde{S}_\ell(B), B)}{c(\alpha k)^{1/2m}}\right)^{\frac{2m}{2m-1}}} \frac{d\alpha}{\alpha} \|f\|_{L^{p_0}(B)}^2 \\ &\leq C_{M,k} \int_{c(2^j r)^{2m}}^{\infty} \left(\frac{r^{2m}}{\alpha}\right)^{2M} \alpha^{2 \times \frac{1}{2m}(\frac{n}{2} - \frac{n}{p_0})} \frac{d\alpha}{\alpha} \|f\|_{L^{p_0}(B)}^2 \\ &\leq C_{M,k} r^{-2n(\frac{1}{p_0} - \frac{1}{2})} 2^{-4mjM} \|f\|_{L^{p_0}(B)}^2. \end{aligned} \quad (3.31)$$

When $\ell = -1, 0$, the same arguments can lead to

$$I_\ell \leq C r^{-2n(\frac{1}{p_0} - \frac{1}{2})} \|f\|_{L^{p_0}(B)}^2.$$

Thus

$$\left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |\mathcal{S}_h^{L,k}(\mathbf{I} - A_{r(B)})f(x)|^2 dx \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}},$$

where $g(j) = C 2^{-j(\frac{n}{2} + \frac{2mM}{2m-1})}$ and $\sum_j g(j) 2^{jn} < \infty$ for $M > \frac{n}{4m}(2m-1)$. We hence prove that $\mathcal{S}_h^{L,k}$ is bounded in $L^p(\mathbb{R}^n)$ with $p_L < p \leq 2$.

(ii) Case where $2 < p < \tilde{p}_L$. Let us first recall a result in [5]:

Lemma 3.5 (Auscher–Hofmann–Martell). *Let $f \in L_{loc}^2(\mathbb{R}_+^{n+1})$. Assume $2 < p < \infty$, then there exists a constant $C = C(n, p) > 0$ such that*

$$\left\| \left(\iint_{\Gamma(x)} |f(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| \left(\int_0^\infty |f(x, t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p}.$$

Note that it is shown in Theorem 3.3 that $g_h^{L,k}$ is bounded on L^p for $2 < p < \tilde{p}_L$. By applying Lemma 3.5, there exists a constant $C > 0$ such that for all $2 < p < \tilde{p}_L$

$$\|S_h^{L,k} f\|_{L^p} \leq C \|g_h^{L,k} f\|_{L^p} \leq C \|f\|_{L^p}.$$

We therefore finish the proof of Theorem 3.4. \square

Remark 3.3. We would like to point out that the L^p boundedness of $S_h^{L,k}$ for $2 < p < \tilde{p}_L$ also can be proved by using the method analogous to the one in the proof of [2, Proposition 6.9] which is based on the good λ inequality introduced by Auscher in [2, Proposition 1.5].

4. Molecular Hardy spaces $H_L^1(\mathbb{R}^n)$

Now we turn to define the molecular Hardy spaces $H_L^1(\mathbb{R}^n)$ associated with the operator L , where L is the homogeneous elliptic operator of order $2m$ in divergence form defined by (1.4)–(1.6).

For a cube Q in \mathbb{R}^n with sides parallel to the axes, here and in the following we denote by $l(Q)$ the side length of Q , by $|Q|$ the volume of Q , and set

$$S_0(Q) = Q, \quad Q_i = 2^i Q, \quad \text{and} \quad S_i(Q) = 2^i Q \setminus 2^{i-1} Q, \quad \text{for } i = 1, 2, \dots,$$

where $2^i Q$ is the cube with the same center as Q and side length $2^i l(Q)$.

In all the following definitions and conclusions, we always assume that $p_L < p < \tilde{p}_L$, where p_L and \tilde{p}_L are defined in (3.8) and (3.9), respectively. Moreover, let $\varepsilon > 0$ and $M \in \mathbb{N}$ with $M > \frac{n}{4m}(2m-1)$.

4.1. Definitions of molecule and molecular Hardy spaces $H_L^1(\mathbb{R}^n)$

Definition 4.1 ((p, ε, M) -molecule). A function u in $L^p \cap L^2$ is called a (p, ε, M) -molecule if $u \in \mathcal{R}(L^M)$, the range of operator L^M , and there exists a cube $Q \subset \mathbb{R}^n$ such that

$$\|u\|_{p,\varepsilon,M,Q} = \sum_{i=0}^{\infty} 2^{i(n-\frac{n}{p}+\varepsilon)} |Q|^{1-\frac{1}{p}} \sum_{k=0}^M \| (l(Q)^{-2m} L^{-1})^k u \|_{L^p(S_i(Q))} \leq 1. \quad (4.1)$$

Remark 4.1. In the expression of (4.1), the operator L^{-1} exists as the unbounded inverse of L since the operator L is one-to-one. In fact, if $Lf = 0$ for some $f \in \mathcal{D}(L)$ which is a dense subset of $W^{m,2}(\mathbb{R}^n)$ (i.e. the domain of the form \mathcal{Q} defined in Section 1), then it follows from Gårding inequality (1.6) that $\|\nabla^m f\| = 0$, which gives $f = 0$ by using the following Gagliardo–Nirenberg type inequality (see e.g. [51, Section 2.3.12])

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla^m f\|_{L^2(\mathbb{R}^n)}^\theta \|f\|_{L^2(\mathbb{R}^n)}^{1-\theta},$$

where $\theta = \frac{n}{2m}(1 - \frac{2}{q})$ and $2 \leq q \leq 2n/(n-2m)$ if $n > 2m$ (resp. $2 \leq q < \infty$ if $n = 2m$ and $2 \leq q \leq \infty$ if $n < 2m$).

Remark 4.2. Similarly to the case of second order operator L (see [42, p. 41]), if u is a (p, ε, M) -molecule adapted to Q , then u is also a classical $(p, 1, 0, \varepsilon)$ -molecule with center at x_0 , the center of Q . Hence, we can easily show that the spaces $H_L^1(\mathbb{R}^n)$ can be embedded into the classical Hardy space $H^1(\mathbb{R}^n)$, but the embedding is not necessary one-to-one. One can see [43] for more details.

Definition 4.2 (*Molecular Hardy space $H_L^1(\mathbb{R}^n)$*). Given a sequence $\{u_j\}$ of (p, ε, M) -molecules, we say that $f = \sum_j \lambda_j u_j$ is a (p, ε, M) -representation of f if $\{\lambda_j\}_{j=1}^\infty \in l^1$, where sum converges in the sense of L^p . Denote

$$H_{L,p,\varepsilon,M}^1(\mathbb{R}^n) = \{f: f \text{ has a } (p, \varepsilon, M)\text{-representation}\}. \quad (4.2)$$

Then the molecular Hardy space $H_L^1(\mathbb{R}^n)$ is defined as the completion of $H_{L,p,\varepsilon,M}^1(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H_{L,p,\varepsilon,M}^1} = \inf \left\{ \sum_{j=0}^\infty |\lambda_j|: f = \sum_{j=0}^\infty \lambda_j u_j \text{ is a } (p, \varepsilon, M)\text{-representation} \right\}. \quad (4.3)$$

In forthcoming subsections, we will use some idea from [42] to show that the molecular Hardy space $H_L^1(\mathbb{R}^n)$ can be characterized by the area integrals. We hence need to introduce another space $\tilde{H}_L^1(\mathbb{R}^n)$ for some technical reasons. Let us start with the δ -representation of a molecule.

4.2. The spaces $H_{L,\delta,p,\varepsilon,M}^1$ and \tilde{H}_L^1

Definition 4.3 (δ -(p, ε, M) representation). For $\delta > 0$, we say that $f = \sum_{j=0}^\infty \lambda_j u_j$ is a δ -(p, ε, M) representation of f if $\{\lambda_j\}_{j=0}^\infty \in l^1$ and each u_j is a (p, ε, M) -molecule adapted to a cube Q_j with its side length at least δ .

Set

$$H_{L,\delta,p,\varepsilon,M}^1(\mathbb{R}^n) = \{f \in L^1: f \text{ has a } \delta\text{-(} p, \varepsilon, M \text{) representation}\}$$

and

$$\hat{H}_L^1(\mathbb{R}^n) = \bigcup_{\delta>0} H_{L,\delta,p,\varepsilon,M}^1(\mathbb{R}^n).$$

Define by $\tilde{H}_L^1(\mathbb{R}^n)$ the completion of $\hat{H}_L^1(\mathbb{R}^n)$ with respect to the following norm:

$$\|f\|_{\tilde{H}_L^1(\mathbb{R}^n)} = \inf \left\{ \sum_{j=0}^\infty |\lambda_j|: f = \sum_{j=0}^\infty \lambda_j u_j \text{ is a } \delta\text{-(} p, \varepsilon, M \text{) representation for some } \delta \right\}. \quad (4.4)$$

Lemma 4.1. With the definitions above, the space $\tilde{H}_L^1(\mathbb{R}^n) = H_L^1(\mathbb{R}^n)$ and the norms are equivalent.

Proof. In fact, since a δ -(p, ε, M) representation clearly converges in $L^p(\mathbb{R}^n)$, by the completeness of H_L^1 , we have trivially that $\tilde{H}_L^1(\mathbb{R}^n) \subseteq H_L^1(\mathbb{R}^n)$. The converse including $H_L^1(\mathbb{R}^n) \subseteq \tilde{H}_L^1(\mathbb{R}^n)$ can be proved by the same idea from [42, p. 49]. \square

5. Characterization of H_L^1 by area integral $\mathcal{S}_h^{L,k}$

In this section, we will characterize H_L^1 by area integral $\mathcal{S}_h^{L,k}$. We first introduce the Hardy spaces $H_{\mathcal{S}_h^{L,k}}^1(\mathbb{R}^n)$ associated with $\mathcal{S}_h^{L,k}$ for $k \in \mathbb{N}$ as follows.

Definition 5.1 (Hardy spaces $H_{\mathcal{S}_h^{L,k}}^1$). Letting $k \in \mathbb{N}$, the space $H_{\mathcal{S}_h^{L,k}}^1$ is defined as the completion of the space $\{f \in L^2(\mathbb{R}^n) : \mathcal{S}_h^{L,k} f \in L^1(\mathbb{R}^n)\}$ with respect to the norm $\|f\|_{H_{\mathcal{S}_h^{L,k}}^1} := \|\mathcal{S}_h^{L,k} f\|_{L^1}$.

Theorem 5.1.

- (a) For given $p \in (p_L, \tilde{p}_L)$, let $f = \sum_{j=0}^{\infty} \lambda_j u_j$ be a (p, ε, M) -representation of f . Then $\sum_{j=0}^{\infty} \lambda_j u_j$ converges in $H_{\mathcal{S}_h^{L,k}}^1(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and $\|\sum_{j=0}^{\infty} \lambda_j u_j\|_{H_{\mathcal{S}_h^{L,k}}^1} \leq C \sum_{j=0}^{\infty} |\lambda_j|$. In particular, we have

$$\|f\|_{H_{\mathcal{S}_h^{L,k}}^1(\mathbb{R}^n)} \leq C \|f\|_{H_L^1(\mathbb{R}^n)}. \quad (5.1)$$

- (b) Let $k \in \mathbb{N}$, if $f \in L^2(\mathbb{R}^n) \cap H_{\mathcal{S}_h^{L,k}}^1(\mathbb{R}^n)$, then there exists a sequence of $f_N \subset \hat{H}_L^1(\mathbb{R}^n)$ such that $\{f_N\}$ converges to f in $\tilde{H}_L^1(\mathbb{R}^n)$. Furthermore, there exist a family of (p, ε, M) -molecules $\{u_j\}_{j=0}^{\infty}$ and a sequence $\{\lambda_j\}_{j=0}^{\infty}$ such that $f = \sum_{j=0}^{\infty} \lambda_j u_j$, with

$$\|f\|_{\tilde{H}_L^1(\mathbb{R}^n)} \leq C \sum_{j=0}^{\infty} |\lambda_j| \leq C \|f\|_{H_{\mathcal{S}_h^{L,k}}^1(\mathbb{R}^n)}. \quad (5.2)$$

Remark 5.1. Note that when $m = 1$ and $k = 1$, Theorem 5.1 returns to Hofmann and Mayboroda's result in [42]. On the other hand, if $2m \geq n$, then the kernel of the semigroup e^{-tL} satisfies Gaussian pointwise estimate (3.4). Thus it follows that from Theorem 5.1 our molecular Hardy space $H_L^1(\mathbb{R}^n)$ is the same one as defined in [28] and [29] by Duong and Yan.

We give below a lemma which will be used in the proof of Theorem 5.1.

Lemma 5.1. Suppose that T is a sublinear operator bounded on $L^p(\mathbb{R}^n)$ for some $p_L < p < \tilde{p}_L$. If there exists $C > 0$ such that for every (p, ε, M) -molecule u

$$\|Tu\|_{L^1(\mathbb{R}^n)} \leq C, \quad (5.3)$$

then T can be extended to a bounded operator from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and

$$\|Tf\|_{L^1(\mathbb{R}^n)} \leq C \|f\|_{H_L^1(\mathbb{R}^n)}.$$

Proof. We may assume that $Tf \geq 0$ for $f \in L^p(\mathbb{R}^n)$. By a density argument it suffices to work with the space $H_{L,p,\varepsilon,M}^1(\mathbb{R}^n)$. Let $f \in H_{L,p,\varepsilon,M}^1(\mathbb{R}^n)$ with a (p, ε, M) -representation: $f = \sum_{j=0}^{\infty} \lambda_j u_j$ such that

$$\|f\|_{H_L^1} \approx \sum_{j=0}^{\infty} |\lambda_j|.$$

Denote $f_k = \sum_{j=0}^k \lambda_j u_j$, then $f_k \rightarrow f$ in $L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. In fact, we see that $f_k \rightarrow f$ in L^p by Definition 4.2. On the other hand, $\|u\|_{L^1(\mathbb{R}^n)} \leq 1$ for every (p, ε, M) -molecule u by the definition of $H_L^1(\mathbb{R}^n)$, thus $\|f - f_k\|_{L^1} \leq \sum_{j=k}^{\infty} |\lambda_j| \rightarrow 0$ as $k \rightarrow \infty$. Since T is bounded on $L^p(\mathbb{R}^n)$, hence $|Tf_k - Tf| \leq T(f - f_k) \rightarrow 0$ in $L^p(\mathbb{R}^n)$ by sublinearity and the assumption of $Tf \geq 0$. On the other hand, when $k > k'$ we have

$$\|Tf_k - Tf_{k'}\|_{L^1} \leq \|T(f_{k'} - f_k)\|_{L^1} \leq C \sum_{j=k'}^k |\lambda_j|.$$

Thus, $\|Tf_k - Tf_{k'}\|_{L^1} \rightarrow 0$ as $k, k' \rightarrow \infty$ and $f_k \rightarrow f$ in $L^1(\mathbb{R}^n)$. Consequently, $\{Tf_k\}$ is a Cauchy sequence in $L^1(\mathbb{R}^n)$, and there exists $g \in L^1(\mathbb{R}^n)$ such that $Tf_k \rightarrow g$ in $L^1(\mathbb{R}^n)$. Thus $Tf = g$ a.e. and $\|Tf\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{H_L^1(\mathbb{R}^n)}$. \square

5.1. Proof of the conclusion (a) in Theorem 5.1

Observe that every (p, ε, M) -molecule is a (q, ε, M) -molecule if $p \geq q > p_L$, so it is enough to consider the case where $p \leq 2$. By Lemma 3.3(ii), $\mathcal{S}_h^{L,k}$ is bounded on L^p for $p_L < p < \tilde{p}_L$ and $k \in \mathbb{N}$. Thus, using Lemma 5.1, it suffices to show that (5.3) is true for $\mathcal{S}_h^{L,k}$. That is, we need to prove that for every (p, ε, M) -molecule u and $k \in \mathbb{N}$, there exists $C > 0$, independent of u , such that $\|\mathcal{S}_h^{L,k} u\|_{L^1(\mathbb{R}^n)} \leq C$. To do this, it is enough to prove

$$\|\mathcal{S}_h^{L,k} (\mathbf{I} - e^{-l(Q)^{2m}L})^M u\|_{L^1(\mathbb{R}^n)} \leq C \quad (5.4)$$

and

$$\|\mathcal{S}_h^{L,k} (\mathbf{I} - (\mathbf{I} - e^{-l(Q)^{2m}L})^M) u\|_{L^1(\mathbb{R}^n)} \leq C. \quad (5.5)$$

We first consider (5.4). Note that

$$\begin{aligned} & \|\mathcal{S}_h^{L,k} (\mathbf{I} - e^{-l(Q)^{2m}L})^M u\|_{L^1(\mathbb{R}^n)} \\ & \leq \sum_{i=0}^{\infty} \|\mathcal{S}_h^{L,k} (\mathbf{I} - e^{-l(Q)^{2m}L})^M (\chi_{S_i(Q)} u)\|_{L^1(\mathbb{R}^n)} \\ & \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (2^{i+j} l(Q))^{n-\frac{n}{p}} \|\mathcal{S}_h^{L,k} (\mathbf{I} - e^{-l(Q)^{2m}L})^M (\chi_{S_i(Q)} u)\|_{L^p(S_j(Q_i))}. \end{aligned} \quad (5.6)$$

For $j = 0, 1, 2$, using the L^p boundedness of $\mathcal{S}_h^{L,k}$ and $(\mathbf{I} - e^{-l(Q)^{2m}L})^M$, we have

$$\|\mathcal{S}_h^{L,k}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)\|_{L^p(S_j(Q_i))} \leq C\|u\|_{L^p(S_i(Q))}.$$

For $j \geq 3$, by Hölder's inequality and (3.28)

$$\begin{aligned} & \|\mathcal{S}_h^{L,k}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)\|_{L^p(S_j(Q_i))}^2 \\ & \leq C|S_j(Q_i)|^{2(\frac{1}{p}-\frac{1}{2})} \int \int_{\Gamma(x)} |(t^{2m}L)^k e^{-t^{2m}L}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)(y)|^2 \frac{dy dt}{t^{n+1}} dx \\ & \leq C|S_j(Q_i)|^{2(\frac{1}{p}-\frac{1}{2})} \iint_{\mathcal{R}(S_j(Q_i))} |(t^{2m}L)^k e^{-t^{2m}L}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)(y)|^2 \frac{dy dt}{t} \\ & \leq C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \int_{\mathbb{R}^n \setminus Q_{i+j-2}} \int_0^\infty |(t^{2m}L)^k e^{-t^{2m}L}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)(y)|^2 \frac{dt}{t} dy \\ & \quad + C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \sum_{\ell=0}^{j-2} \int_{S_\ell(Q_i)} \int_{(2^{j-1-2^\ell})^{2i}l(Q)}^\infty |(t^{2m}L)^k e^{-t^{2m}L} \\ & \quad \times (\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)(y)|^2 \frac{dt}{t} dy \\ & := I + \sum_{\ell=0}^{j-2} I_\ell. \end{aligned}$$

Using an idea analogous to the proof of (3.30) and (3.31), we may get the estimates of I and I_ℓ ($\ell = 0, 1, \dots, j-2$). In fact, for I , by (3.13) we have

$$\begin{aligned} I &= C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \|\mathcal{S}_h^{L,k}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)\|_{L^2(\mathbb{R}^n \setminus Q_{i+j-2})}^2 \\ &\leq C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} (l(Q))^{2(-\frac{n}{p}+\frac{n}{2})} \left(\frac{l(Q)^{2m}}{d(S_i(Q), \mathbb{R}^n \setminus Q_{i+j-2})^{2m}} \right)^{\frac{2M}{2m-1}} \|u\|_{L^p(S_i(Q))}^2 \\ &\leq C(2^{i+j})^{-\frac{4mM}{2m-1}+2(\frac{n}{p}-\frac{n}{2})} \|u\|_{L^p(S_i(Q))}^2. \end{aligned} \quad (5.7)$$

As for I_ℓ ($\ell = 0, 1, \dots, j-2$), letting $s = \frac{t^{2m}}{M+1}$, we have

$$\begin{aligned} I_\ell &\leq C_{M,k} (2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \\ &\quad \times \int_{c(2^{i+j}l(Q))^{2m}}^\infty \|(sL)^k e^{-s(M+1)L}(\mathbf{I} - e^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}u)\|_{L^2(S_\ell(Q_i))}^2 \frac{ds}{s} \end{aligned}$$

$$\leq C_{M,k} (2^{i+j} l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \int_{c(2^{i+j} l(Q))^{2m}}^{\infty} \left(\frac{l(Q)^{2m}}{s} \right)^{2M} \\ \times \left\| \left(\frac{s}{l(Q)^{2m}} (e^{-sL} - e^{-(l(Q)^{2m}+s)L}) \right)^M (sL)^k e^{-sL} (\chi_{S_i(Q)} u) \right\|_{L^2(S_\ell(Q_i))}^2 \frac{ds}{s}.$$

By the L^2 off-diagonal estimates for $\frac{s}{l(Q)^{2m}} (e^{-sL} - e^{-(l(Q)^{2m}+s)L})$ (see (iii) of Theorem 3.2) and L^p – L^2 off-diagonal estimates for $(sL)^k e^{-sL}$ (see (iv) of Theorem 3.2), and use Lemma 3.2 we get

$$I_\ell \leq C (2^{i+j} l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \int_{c(2^{i+j} l(Q))^{2m}}^{\infty} \left(\frac{l(Q)^{2m}}{s} \right)^{2M} \\ \times s^{\frac{1}{m}(\frac{n}{2}-\frac{n}{p})} e^{-2(M+1)(\frac{d(S_\ell(Q_i), S_i(Q))}{cs^{1/2m}})^{\frac{2m}{2m-1}}} \frac{ds}{s} \|u\|_{L^p(S_i(Q))}^2 \\ \leq C (2^{i+j} l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \int_{c(2^{i+j} l(Q))^{2m}}^{\infty} \left(\frac{l(Q)^{2m}}{s} \right)^{2M} s^{2 \times \frac{1}{2m}(\frac{n}{2}-\frac{n}{p})} \frac{ds}{s} \|u\|_{L^p(S_i(Q))}^2 \\ \leq C (2^{i+j})^{-4mM} \|u\|_{L^p(S_i(Q))}^2. \quad (5.8)$$

Thus, by (5.6)–(5.8) we obtain (5.4):

$$\|S_h^{L,k} (\mathbf{I} - e^{-l(Q)^{2m}L})^M u\|_{L^1(\mathbb{R}^n)} \leq C \sum_{i=0}^{\infty} \left\{ \sum_{j=3}^{\infty} ([2^{-(i+j)(\frac{2mM}{2m-1} - (\frac{n}{p}-\frac{n}{2}))} \right. \\ \left. + \sqrt{j} 2^{-(i+j)(2mM - (\frac{n}{p}-\frac{n}{2}))}] (2^{i+j} l(Q))^{n-\frac{n}{p}} \right. \\ \left. + (2^i l(Q))^{n-\frac{n}{p}} \right\} \|u\|_{L^p(S_i(Q))} \leq C.$$

Let us now turn to (5.5). To get (5.5), it is sufficient to show that

$$\sup_{1 \leq \ell \leq M} \left\| S_h^{L,k} \left(\frac{\ell}{M} l(Q)^{2m} L e^{-\frac{\ell}{M} l(Q)^{2m} L} \right)^M ((l(Q)^{2m} L)^{-M} u) \right\|_{L^1(\mathbb{R}^n)} \leq C. \quad (5.9)$$

Like (5.6), we have

$$\|S_h^{L,k} (l(Q)^{2m} L e^{-l(Q)^{2m} L})^M ((l(Q)^{2m} L)^{-M} u)\|_{L^1(\mathbb{R}^n)}^2 \\ \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (2^{i+j} l(Q))^{n-\frac{n}{p}} \|S_h^{L,k} (l(Q)^{2m} L e^{-l(Q)^{2m} L})^M \\ \times (\chi_{S_i(Q)} (l(Q)^{2m} L)^{-M} u)\|_{L^p(S_j(Q_i))}^2. \quad (5.10)$$

For $0 \leq j \leq 2$, using the L^p boundedness of $S_h^{L,k}$ and $(tLe^{-tL})^M$, we have

$$\begin{aligned} & \|S_h^{L,k}(l(Q)^{2m}Le^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}(l(Q)^{2m}L)^{-M}u)\|_{L^p(S_j(Q_i))} \\ & \leq C\|(l(Q)^{2m}L)^{-M}u\|_{L^p(S_i(Q))}. \end{aligned}$$

For $j \geq 3$, by the definition of $S_h^{L,k}$, Hölder's inequality and (3.28), we have

$$\begin{aligned} & \|S_h^{L,k}(l(Q)^{2m}Le^{-l(Q)^{2m}L})^M(\chi_{S_i(Q)}(l(Q)^{2m}L)^{-M}u)\|_{L^p(S_j(Q_i))}^2 \\ & \leq C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \int_{\mathbb{R}^n \setminus Q_{i+j-2}} \int_0^\infty |(t^{2m}L)^k e^{-t^{2m}L}(l(Q)^{2m}Le^{-l(Q)^{2m}L})^M \\ & \quad \times (\chi_{S_i(Q)}(l(Q)^{2m}L)^{-M}u)|^2 \frac{dt}{t} dy \\ & \quad + C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \sum_{\ell=0}^{j-2} \int_{S_\ell(Q_i)} \int_{(2^{j-1}-2^\ell)2^i l(Q)}^\infty |(t^{2m}L)^k e^{-t^{2m}L}(l(Q)^{2m}Le^{-l(Q)^{2m}L})^M \\ & \quad \times (\chi_{S_i(Q)}(l(Q)^{2m}L)^{-M}u)|^2 \frac{dt}{t} dy \\ & := \tilde{I} + \sum_{\ell=0}^{j-2} \tilde{I}_\ell. \end{aligned}$$

For \tilde{I} , we use (3.14) instead of (3.13) in (5.7) to get

$$\tilde{I} \leq C(2^{i+j})^{-\frac{4mM}{2m-1}+2(\frac{n}{p}-\frac{n}{2})} \|(l(Q)^{2m}L)^{-M}u\|_{L^p(S_i(Q))}^2.$$

To deal with \tilde{I}_ℓ , using the same argument as for I_ℓ we have

$$\begin{aligned} \tilde{I}_\ell & \leq C(2^{i+j}l(Q))^{2(\frac{n}{p}-\frac{n}{2})} \int_{c(2^{i+j}l(Q))^{2m}}^\infty \left(\frac{l(Q)^{2m}}{s}\right)^{2M} \\ & \quad \times \|(sL)^k e^{-sL}(sLe^{-l(Q)^{2m}+s}L)^M(\chi_{S_i(Q)}(l(Q)^{2m}L)^{-M}u)\|_{L^2(S_\ell(Q_i))}^2 \frac{ds}{s} \\ & \leq C(2^{i+j})^{-4mM} \|(l(Q)^{2m}L)^{-M}u\|_{L^p(S_i(Q))}^2. \end{aligned}$$

Thus, with the above estimates and the definition of a (p, ε, M) -molecule, we get (5.9) and (5.5) follows. We hence complete the proof of (a) in Theorem 5.1.

5.2. Proof of the conclusion (b) in Theorem 5.1

We shall use an idea from [42]. The proof of (b) will be completed by dividing into some steps as follows.

Step 1. Let $k \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^n) \cap H_{S_h}^{1,L,k}(\mathbb{R}^n)$. By the L^2 functional calculus (see [52]), we can write

$$\begin{aligned} f &= C_{M,m,k} \int_0^\infty ((t^{2m}L)^k e^{-t^{2m}L})^{M+2} f \frac{dt}{t} \\ &= C_{M,m,k} \lim_{N \rightarrow \infty} \int_{\frac{1}{N}}^N ((t^{2m}L)^k e^{-t^{2m}L})^{M+2} f \frac{dt}{t} \\ &:= \lim_{N \rightarrow \infty} f_N, \end{aligned}$$

where the limit is understood in the L^2 sense.

Step 2. We show that, for fixed N , f_N can be decomposed into a δ -representation with $\delta \approx \frac{1}{N}$. To do this, for $\ell \in \mathbb{Z}$, we define $O_\ell := \{x \in \mathbb{R}^n: S_h^{L,k} f(x) > 2^\ell\}$ and $O_\ell^* := \{x \in \mathbb{R}^n: \mathcal{M}(\chi_{O_\ell}) > 1 - \gamma\}$ for some fixed $0 < \gamma < 1$. Then $O_\ell \subset O_\ell^*$ and $|O_\ell^*| \leq C(\gamma)|O_\ell|$ for every ℓ . Let $\{Q_\ell^j\}_{j \in \mathbb{N}}$ be the Whitney decomposition of O_ℓ^* and

$$\widehat{O}_\ell^* := \{(x, t) \in \mathbb{R}^n \times (0, \infty): d(x, (O_\ell^*)^c) \geq t\}$$

the tent region of O_ℓ^* . For every $\ell \in \mathbb{Z}$, $j \in \mathbb{N}$, set

$$T_\ell^j := (Q_\ell^j \times (0, \infty)) \cap \widehat{O}_\ell^* \cap (\widehat{O}_{\ell+1}^*)^c. \quad (5.11)$$

Then, we have

$$\begin{aligned} f_N &= C_{M,m,k} \sum_{\ell \in \mathbb{Z}, j \in \mathbb{N}} \int_{\frac{1}{N}}^N ((t^{2m}L)^k e^{-t^{2m}L})^{M+1} (\chi_{T_\ell^j} (t^{2m}L)^k e^{-t^{2m}L}) f \frac{dt}{t} \\ &:= C_{M,m,k} \sum_{\ell \in \mathbb{Z}, j \in \mathbb{N}} \lambda_\ell^j u_{\ell,N}^j \end{aligned} \quad (5.12)$$

where $\lambda_\ell^j = 2^\ell |Q_\ell^j|$ and

$$u_{\ell,N}^j = \frac{1}{\lambda_\ell^j} \int_{\frac{1}{N}}^N ((t^{2m}L)^k e^{-t^{2m}L})^{M+1} (\chi_{T_\ell^j} (t^{2m}L)^k e^{-t^{2m}L}) f \frac{dt}{t}.$$

Note that

$$\sum_{\ell \in \mathbb{Z}, j \in \mathbb{N}} \lambda_{\ell}^j = \sum_{\ell \in \mathbb{Z}} 2^{\ell} \sum_{j \in \mathbb{N}} |Q_{\ell}^j| \leq C \sum_{\ell \in \mathbb{Z}} 2^{\ell} |O_{\ell}^*| \leq C \sum_{\ell \in \mathbb{Z}} 2^{\ell} |O_{\ell}| \leq C \|S_h^{L,k} f\|_{L^1}. \quad (5.13)$$

Hence, if we can show that there exists some $C > 0$ such that for every $\ell \in \mathbb{Z}$, $j \in \mathbb{N}$, the function $C^{-1}u_{\ell,N}^j$ is a (p, ε, M) -molecule associated with the cube Q_{ℓ}^j , then f_N is a δ -(p, ε, M) representation with $\delta \approx \frac{1}{N}$ by (5.12).

Step 3. We now show that for each $\ell \in \mathbb{Z}$, $j \in \mathbb{N}$, $u_{\ell,N}^j$ is a (p, ε, M) -molecule associated with the cube Q_{ℓ}^j up to a constant. By Definition 4.1 of (p, ε, M) -molecule, we need to prove that there exists a $C > 0$, independent of j, ℓ, N , such that

$$\sum_{i=0}^{\infty} 2^{i(n-\frac{n}{p}+\varepsilon)} |Q_{\ell}^j|^{1-\frac{1}{p}} \sum_{v=0}^M \| (l(Q_{\ell}^j)^{-2m} L^{-1})^v u_{\ell,N}^j \|_{L^p(S_i(Q_{\ell}^j))} \leq C. \quad (5.14)$$

We only prove the situation of $2 \leq p < \tilde{p}_L$, the case $p \leq 2$ is similar. We first consider $\|u_{\ell,N}^j\|_{L^p(S_i(Q_{\ell}^j))}$. Choose $h \in L^{p'}(S_i(Q_{\ell}^j))$ such that $\|h\|_{L^{p'}(S_i(Q_{\ell}^j))} = 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $i \in \mathbb{N} \cup \{0\}$, then we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} u_{\ell,N}^j(x) \overline{h(x)} dx \right| \\ &= \frac{1}{\lambda_{\ell}^j} \left| \int_{\mathbb{R}^n} \int_{\frac{1}{N}}^N \chi_{T_{\ell}^j}(t^{2m} L)^k e^{-t^{2m} L} f(x) \overline{((t^{2m} L)^k e^{-t^{2m} L})^{M+1} h(x)} \frac{dx dt}{t} \right| \\ &\leq \frac{1}{\lambda_{\ell}^j} \iint_{T_{\ell}^j} |(t^{2m} L)^k e^{-t^{2m} L} f(x) ((t^{2m} L)^k e^{-t^{2m} L})^{M+1} h(x)| \frac{dx dt}{t} \\ &= \frac{1}{\lambda_k^j} \iint_{(\widehat{O_{\ell+1}^*})^c} \chi_{\ell}^j(x, t) |(t^{2m} L)^k e^{-t^{2m} L} f(x) ((t^{2m} L)^k e^{-t^{2m} L})^{M+1} h(x)| \frac{dx dt}{t}, \end{aligned}$$

where $\chi_{\ell}^j := \chi_{(Q_{\ell}^j \times (0, \infty)) \cap \widehat{O_{\ell}^*}}$. Applying (3.27) in Lemma 3.4 with $F = O_{\ell+1}^c$, $F^{\otimes} = (O_{\ell+1}^*)^c$, $\mathcal{R}^1(F^{\otimes}) = (\widehat{O_{\ell+1}^*})^c$ and

$$\Phi(x, t) = \chi_{\ell}^j(x, t) |(t^{2m} L)^k e^{-t^{2m} L} f(x) ((t^{2m} L)^k e^{-t^{2m} L})^{M+1} h(x)| t^{-n-1},$$

we get

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} u_{\ell,N}^j(x) \overline{h(x)} dx \right| \\
& \leq \frac{C}{\lambda_\ell^j} \int \iint_{O_{\ell+1}^c \cap \Gamma(x)} \chi_\ell^j(y,t) |(t^{2m}L)^k e^{-t^{2m}L} f(y) (((t^{2m}L)^k e^{-t^{2m}L})^{M+1})^* h(y)| \frac{dy dt}{t^{n+1}} dx \\
& \leq \frac{C}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \left(\iint_{\Gamma(x)} |(t^{2m}L)^k e^{-t^{2m}L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \left(\iint_{\Gamma(x) \cap (Q_\ell^j \times (0,\infty)) \cap \widehat{O}_\ell^*} |(((t^{2m}L)^k e^{-t^{2m}L})^{M+1})^* h(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{p'}{2}} dx \right)^{\frac{1}{p'}} \\
& := I_1 \times I_2,
\end{aligned}$$

where the last inequality is based on Hölder's inequality and the fact that whenever $(y, t) \in \Gamma(x) \cap (Q_\ell^j \times (0, \infty)) \cap \widehat{O}_\ell^*$, then $x \in cQ_\ell^j$, where the constant c is related to the constant of Whitney decomposition, without loss of generality, we may assume that $c \leq 3$.

For I_1 , we observe that

$$I_1 \leq \frac{C}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} (\mathcal{S}_h^{L,k} f(x))^p dx \right)^{\frac{1}{p}}$$

and by the definition of $O_{\ell+1}$, we have

$$I_1 \leq \frac{C}{\lambda_\ell^j} 2^{\ell+1} |Q_\ell^j|^{\frac{1}{p}} \leq C |Q_\ell^j|^{\frac{1}{p}-1}.$$

Let us now deal with I_2 . We first consider $i \leq 4$. By the L^p boundedness of $\mathcal{S}_h^{L,k(M+1)}$ (see Theorem 3.4), we have

$$I_2 \leq C \|\mathcal{S}_h^{L,k(M+1)} h\|_{L^{p'}(\mathbb{R}^n)} \leq C \|h\|_{L^{p'}(S_i(Q_\ell^j))} \leq C.$$

When $i \geq 5$, we involve Hölder's inequality to get that

$$\begin{aligned}
I_2 & \leq C |Q_\ell^j|^{\frac{1}{p'} - \frac{1}{2}} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \iint_{(Q_\ell^j \times (0,\infty)) \cap \widehat{O}_\ell^*} |(((t^{2m}L)^k e^{-t^{2m}L})^{M+1})^* h(y)|^2 \frac{dy dt}{t^{n+1}} dx \right)^{\frac{1}{2}} \\
& \leq C |Q_\ell^j|^{\frac{1}{p'} - \frac{1}{2}} \left(\int_{3cQ_\ell^j} \int_0^{cl(Q_\ell^j)} |(((t^{2m}L)^k e^{-t^{2m}L})^{M+1})^* h(y)|^2 \frac{dt}{t} dy \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C |Q_\ell^j|^{\frac{1}{p'} - \frac{1}{2}} \left(\int_0^{cl(Q_\ell^j)} \|((t^{2m}L)^k e^{-t^{2m}L})^{M+1} * h(y)\|_{L^2(3cQ_\ell^j)}^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \quad (5.15)$$

where in the second inequality above, we use (3.28) in Lemma 3.4 for $F = O_{\ell+1}^c \cap 3cQ_\ell^j$ and

$$\Phi(y, t) = \chi_\ell^j(y, t) |((t^{2m}L)^k e^{-t^{2m}L})^{M+1} * h(y)|^2 t^{-n-1}.$$

Applying (iv) of Theorem 3.2 to (5.15), we have

$$\begin{aligned} I_2 &\leq C |Q_\ell^j|^{\frac{1}{p'} - \frac{1}{2}} \left(\int_0^{cl(Q_\ell^j)} e^{-\left(\frac{d(3cQ_\ell^j, S_i(Q_\ell^j))}{t}\right)^{\frac{2m}{2m-1}}} t^{2n(\frac{1}{2} - \frac{1}{p'})} \frac{dt}{t} \right)^{\frac{1}{2}} \|h\|_{L^{p'}(S_i(Q_\ell^j))} \\ &\leq C |Q_\ell^j|^{\frac{1}{p'} - \frac{1}{2}} \left(\int_0^{cl(Q_\ell^j)} \left(\frac{t}{2^i l(Q_\ell^j)}\right)^{2(\frac{n}{p'} + \varepsilon')} t^{2n(\frac{1}{2} - \frac{1}{p'})} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq C 2^{-i(n - \frac{n}{p} + \varepsilon')}, \end{aligned} \quad (5.16)$$

where we may choose $\varepsilon' > \varepsilon$. We hence have

$$\sum_{i=0}^{\infty} 2^{i(n - \frac{n}{p} + \varepsilon)} |Q_\ell^j|^{1 - \frac{1}{p}} \|u_{\ell, N}^j\|_{L^p(S_i(Q_\ell^j))} \leq C. \quad (5.17)$$

Now we turn to estimate $\|(l(Q)^{-2m}L^{-1})^v u_\ell^j\|_{L^p(S_i(Q_k^j))}$ for $1 \leq v \leq M$. It is quite similar to the previous situation, we only write down the main steps. For $h \in L^{p'}(S_i(Q_\ell^j))$, we get

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (l(Q)^{-2m}L^{-1})^v u_{\ell, N}^j(x) \overline{h(x)} dx \right| \\ &\leq \frac{C}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \left(\iint_{\Gamma(x)} |(t^{2m}L)^k e^{-t^{2m}L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \left(\iint_{\Gamma(x)} \chi_\ell^j \left| \left(\frac{t}{l(Q)} \right) \right|^{2mv} \right. \right. \\ &\quad \times \left. \left. ((t^{2m}L)^{k(M+1)-v} e^{-(M+1)t^{2m}L})^* h(y) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{p'}{2}} dx \Big)^{\frac{1}{p'}} \\ &:= \tilde{I}_1 \times \tilde{I}_2. \end{aligned}$$

Observe that \tilde{I}_1 equals to I_1 . For \tilde{I}_2 , by the same argument along the lines (5.15)–(5.16) we can get $\tilde{I}_2 \leq C 2^{-i(n-\frac{n}{p}+\varepsilon')}$, where $\varepsilon' > \varepsilon$. Thus, for $1 \leq v \leq M$

$$\sum_{i=0}^{\infty} 2^{i(n-\frac{n}{p}+\varepsilon)} |Q_\ell^j|^{1-\frac{1}{p}} \| (l(Q)^{-2m} L^{-1})^v u_{\ell,N}^j \|_{L^p(S_i(Q_\ell^j))} \leq C. \quad (5.18)$$

Thus, (5.14) follows from (5.17) and (5.18).

Remark 5.2. From the proof above, it is easy to see that (5.17) and (5.18) hold uniformly in N . In particular, we have

$$\sup_N \|u_{\ell,N}^j\|_{L^p(\mathbb{R}^n)} \leq C |Q_\ell^j|^{\frac{1}{p}-1}$$

and

$$\sup_N \| (l(Q)^{-2m} L^{-1})^v u_{\ell,N}^j \|_{L^p(\mathbb{R}^n)} \leq C |Q_\ell^j|^{\frac{1}{p}-1} \quad \text{for } 1 \leq v \leq M.$$

Therefore, by (5.13), we get immediately

$$\sup_N \|f_N\|_{\tilde{H}_L^1(\mathbb{R}^n)} \leq C \sum_{\ell \in \mathbb{Z}, j \in \mathbb{N}} \lambda_\ell^j \leq C \|S_h^{L,k} f\|_{L^1(\mathbb{R}^n)}.$$

Step 4. Now we prove that $f_N \rightarrow f$ in $\tilde{H}_L^1(\mathbb{R}^n)$ as $N \rightarrow \infty$.

By (5.14) we see that $f_N \in \hat{H}_L^1(\mathbb{R}^n)$ for fixed N . Below we first show that for fixed j, ℓ , $\{u_{\ell,N}^j\}$ converges weakly in $L^p(\mathbb{R}^n)$ for $p_L < p < \tilde{p}_L$ as $N \rightarrow \infty$, and the limit is also a (p, ε, M) -molecule.

For any $h \in L^2(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$, by the L^2 boundedness of $g_h^{L,k}$ and $S_h^{L,k(M+1)}$ (see (3.17) and Theorem 3.4, respectively) and the dominated convergence theorem, as $N \rightarrow \infty$, we get

$$\begin{aligned} \langle u_{\ell,N}^j, h \rangle &= \frac{1}{\lambda_\ell^j} \int_{\mathbb{R}^n} \int_{\frac{1}{N}}^N \chi_{T_\ell^j}(t^{2m} L)^k e^{-t^{2m} L} f(x) \overline{((t^{2m} L)^k e^{-t^{2m} L})^{M+1} *} h(x)} \frac{dt}{t} dx \\ &\rightarrow \frac{1}{\lambda_\ell^j} \int_{\mathbb{R}^n} \int_0^\infty \chi_{T_\ell^j}(t^{2m} L)^k e^{-t^{2m} L} f(x) \overline{((t^{2m} L)^k e^{-t^{2m} L})^{M+1} *} h(x)} \frac{dt}{t} dx. \end{aligned} \quad (5.19)$$

Similarly, the same conclusion is also true for $(l(Q)^{-2m} L^{-1})^v u_{\ell,N}^j$ and $1 \leq v \leq M$.

Note that we have proved that $C^{-1} u_{\ell,N}^j$ is a molecule with adapted cube Q_ℓ^j and (5.14) holds uniformly in N for $u_{\ell,N}^j$. Thus, $\{u_{\ell,N}^j\}$ has a weak limit u_ℓ^j in $L^p(\mathbb{R}^n)$, and $|\langle u_\ell^j, h \rangle| \leq C |Q_\ell^j|^{\frac{1}{p}-1} \|h\|_{L^{p'}(\mathbb{R}^n)}$. Therefore, $u_k^j \in L^p(\mathbb{R}^n)$ by the Riesz representation theorem.

On the other hand, it is easy to check that $C^{-1}u_\ell^j$ is also a molecule by using the uniform $L^p(S_i(Q_\ell^j))$ bounds for $\{u_{\ell,N}^j\}$ and $\{(l(Q)^{-2m}L^{-1})^v u_{\ell,N}^j\}$ ($1 \leq v \leq M$) (see Remark 5.2).

Next, we need to prove that $\{f_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $\tilde{H}_L^1(\mathbb{R}^n)$. Recall $f_N = \sum \lambda_\ell^j u_{\ell,N}^j$. For any $K \in \mathbb{N}$, we may write

$$f_N = \sum_{j+\ell \leq K} \lambda_\ell^j u_{\ell,N}^j + \sum_{j+\ell > K} \lambda_\ell^j u_{\ell,N}^j =: \sigma_K(N) + R_K(N). \quad (5.20)$$

Notice that as $K \rightarrow \infty$,

$$\sup_N \|R_K(N)\|_{\tilde{H}_L^1(\mathbb{R}^n)} \leq \sum_{j+\ell > K} |\lambda_\ell^j| \rightarrow 0.$$

Thus, for given $\eta > 0$, we may choose K in (5.20) such that $\sup_N \|R_K(N)\|_{\tilde{H}_L^1(\mathbb{R}^n)} < \eta$. Hence we only need to estimate

$$\|\sigma_K(N) - \sigma_K(N')\|_{\tilde{H}_L^1(\mathbb{R}^n)} = \left\| \sum_{j+\ell \leq K} \lambda_\ell^j (u_{\ell,N}^j - u_{\ell,N'}^j) \right\|_{\tilde{H}_L^1(\mathbb{R}^n)} \quad \text{for } N, N' \in \mathbb{N}.$$

To this end, it is sufficient to show that for any given $\epsilon > 0$ and $p_L < p < \tilde{p}_L$, $M > \frac{n}{4m}(2m-1)$, and fixed $K \in \mathbb{N}$, there exists an integer $\tilde{N} = \tilde{N}(\epsilon, p, \epsilon, M, K)$ such that for $N \geq N' \geq \tilde{N}$

$$\max_{j+\ell \leq K} \|u_{\ell,N}^j - u_{\ell,N'}^j\|_{p, \epsilon, M, Q_\ell^j} < \epsilon. \quad (5.21)$$

It is easy to see that, if \tilde{N} is large enough and $N \geq N' \geq \tilde{N}$

$$\int_{N'}^N ((t^{2m}L)^k e^{-t^{2m}L})^{M+1} (\chi_{T_\ell^j}(t^{2m}L)^k e^{-t^{2m}L}) f \frac{dt}{t} \equiv 0.$$

Hence, if we denote $\mu_{\ell,N,N'}^j = u_{\ell,N}^j - u_{\ell,N'}^j$, then

$$\mu_{\ell,N,N'}^j = \frac{1}{\lambda_\ell^j} \int_{\frac{1}{N}}^{\frac{1}{N'}} ((t^{2m}L)^k e^{-t^{2m}L})^{M+1} (\chi_{T_\ell^j}(t^{2m}L)^k e^{-t^{2m}L}) f \frac{dt}{t}.$$

Thus, for $\epsilon' > \epsilon$, $2 \leq q < \tilde{p}_L$ and $t < cl(Q_\ell^j)$, if taking $h \in L^{q'}(S_i(Q_\ell^j))$ with $\|h\|_{L^{q'}(S_i(Q_\ell^j))} = 1$, then by the same argument from (5.15) to (5.16), we get

$$\begin{aligned}
|\langle \mu_{\ell,N,N'}^j, h \rangle| &= \left| \frac{1}{\lambda_\ell^j} \int_{\mathbb{R}^n} \int_{\frac{1}{N'}}^{\frac{1}{N}} \chi_{T_\ell^j} (t^{2m} L)^k e^{-t^{2m} L} f(x) \overline{((t^{2m} L)^k e^{-t^{2m} L})^{M+1} *} h(x)} \frac{dt}{t} dx \right| \\
&\leq C \frac{1}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} (S_{h,N'}^{L,k} f(x))^q dx \right)^{\frac{1}{q}} 2^{-i(n-\frac{n}{q}+\varepsilon')}, \tag{5.22}
\end{aligned}$$

where

$$S_{h,N'}^{L,k} f(x) := \left(\iint_{|x-y| < t < \frac{1}{N'}} |(t^{2m} L)^k e^{-t^{2m} L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Notice that $f \in L^2(\mathbb{R}^n)$ and $S_{h,N'}^{L,k} f \rightarrow 0$ in $L^2(\mathbb{R}^n)$ as $N' \rightarrow \infty$ by the dominated convergence theorem. We may choose \tilde{N} large sufficiently, such that for $N' > \tilde{N}$

$$\|S_{h,N'}^{L,k} f\|_{L^2(\mathbb{R}^n)} \leq C \epsilon^\alpha \min_{j+\ell \leq K} \lambda_\ell^j |Q_\ell^j|^{-\frac{1}{2}}, \tag{5.23}$$

where the choice of α depends on q in (5.22).

When $p_L < p \leq 2$, letting $q = 2$ in (5.22) and $\alpha = 1$ in (5.23) we have

$$\|\mu_{\ell,N,N'}^j\|_{L^2(S_i(Q_\ell^j))} \leq \epsilon 2^{-i(\frac{n}{2}+\varepsilon')} |Q_\ell^j|^{-\frac{1}{2}}.$$

By Hölder's inequality, we get

$$\begin{aligned}
\|\mu_{\ell,N,N'}^j\|_{L^p(S_i(Q_\ell^j))} &\leq \|\mu_{\ell,N,N'}^j\|_{L^2(S_i(Q_\ell^j))} |S_i(Q_\ell^j)|^{\frac{1}{p}-\frac{1}{2}} \\
&\leq \epsilon 2^{-i(n-\frac{n}{p}+\varepsilon')} |Q_\ell^j|^{\frac{1}{p}-1}. \tag{5.24}
\end{aligned}$$

When $2 < p < \tilde{p}_L$, we choose $p < r < \tilde{p}_L$, notice that $S_{h,N'}^{L,k} f \leq 2^{\ell+1}$ on $O_{\ell+1}^c$, then

$$\left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} (S_{h,N'}^{L,k} f(x))^r dx \right)^{\frac{1}{r}} \leq C 2^\ell |Q_\ell^j|^{\frac{1}{r}}.$$

Thus we have

$$\begin{aligned}
\|\mu_{\ell,N,N'}^j\|_{L^p(S_i(Q_\ell^j))} &\leq \|\mu_{\ell,N,N'}^j\|_{L^2(S_i(Q_\ell^j))}^\theta \|\mu_{\ell,N,N'}^j\|_{L^r(S_i(Q_\ell^j))}^{1-\theta} \\
&\leq \epsilon 2^{-i(n-\frac{n}{p}+\varepsilon')} |Q_\ell^j|^{\frac{1}{p}-1}, \tag{5.25}
\end{aligned}$$

where $0 < \theta < 1$ with $\frac{\theta}{2} + \frac{1-\theta}{r} = \frac{1}{p}$ and choosing $\alpha = \frac{1}{\theta}$ in (5.23).

By repeating the previous argument of dealing with $\mu_{\ell,N,N'}^j$ for $(l(Q)^{-2m}L^{-1})^v \mu_{\ell,N,N'}^j$ ($1 \leq v \leq M$), and combining (5.24) and (5.25) we conclude (5.21). Thus $\{f_N\}$ is a Cauchy sequence in $\tilde{H}_L^1(\mathbb{R}^n)$.

Step 5. Finally, we show that $f_N \rightarrow f$ in $H_{S_h}^{1,k}(\mathbb{R}^n)$ as $N \rightarrow \infty$.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, by the fact that $u_{\ell,N}^j \rightarrow u_\ell^j$ weakly in $L^p(\mathbb{R}^n)$ and the absolute convergence we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\sum \lambda_\ell^j u_\ell^j \right) \varphi dx &= \sum \lambda_\ell^j \lim_{N \rightarrow \infty} \int_{\mathbb{R}^n} u_{\ell,N}^j \varphi dx \\ &= \lim_{N \rightarrow \infty} \sum \int_{\mathbb{R}^n} \lambda_k^j u_{\ell,N}^j \varphi dx \\ &= \lim_{N \rightarrow \infty} \int_{\mathbb{R}^n} \left(\sum \lambda_\ell^j u_{\ell,N}^j \right) \varphi dx \\ &= \lim_{N \rightarrow \infty} \int_{\mathbb{R}^n} f_N \varphi dx \\ &= \int_{\mathbb{R}^n} f \varphi dx. \end{aligned} \quad (5.26)$$

Thus we conclude that $f = \sum \lambda_\ell^j u_\ell^j$ almost everywhere. Combining the fact that f_N is a Cauchy sequence in $\tilde{H}_L^1(\mathbb{R}^n)$, we finally proved that $f_N \rightarrow f$ in $\tilde{H}_L^1(\mathbb{R}^n)$, then by using (5.1) which we have proved in Section 5.1 to get that $f_N \rightarrow f$ in $H_{S_h}^{1,k}(\mathbb{R}^n)$, which finishes the proof of (b) in Theorem 5.1.

From the proof of Theorem 5.1, we establish the following facts:

Corollary 5.1. *The space $H_L^1(\mathbb{R}^n) = H_{S_h}^{1,k}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and the norms are equivalent. In particular, the spaces $H_L^1(\mathbb{R}^n)$ coincide for different choices of $p_L < p < \tilde{p}_L$, $\varepsilon > 0$ and $M \in \mathbb{N}$, $M > \frac{n}{4m}(2m-1)$.*

Remark 5.3. By Definition 4.1, if u is a (p, ε, M) -molecule, then

$$\| (l(Q)^{-2m}L^{-1})^v u \|_{L^p(S_i(Q))} \leq 2^{-i(n-\frac{n}{p}+\varepsilon)} |Q|^{\frac{1}{p}-1}, \quad v = 0, 1, \dots, M; \quad i = 1, 2, \dots \quad (5.27)$$

Conversely, if $u \in L^p \cap L^2$ satisfies the conditions (5.27), then u also is a (p, ε', M) -molecule for some $\varepsilon' > \varepsilon$. We remark that Hardy space $H_L^1(\mathbb{R}^n)$ is independent of the choices of p, ε, M although a molecule belongs to some specific (p, ε, M) -molecule class (see Corollary 5.1). Therefore, in this sense, the condition (4.1) in the definition of (p, ε, M) -molecule may be replaced by (5.27) (see also (1.9) and (1.10) in [42]).

6. Characterization of H_L^1 by area integral $\mathcal{S}_P^{L,k}$

As shown in the introduction, the classical Hardy space $H^1(\mathbb{R}^n)$ can be characterized by the square function associated to the Poisson semigroup $e^{-t\sqrt{-\Delta}}$, which is the operator semigroup of the solution of the Laplace equation $\partial_{tt}u + \Delta u = 0$ on the upper space of $\mathbb{R}_+^{n+1} = \{(t, x) \mid x \in \mathbb{R}^n, t > 0\}$. Now we want to give a similar result for the higher order divergence form elliptic operator L . By functional calculus we know that the semigroup $e^{-t\sqrt{L}}$ is a solution operator of the higher order equation $\partial_{tt}u = Lu$. Although the study of such higher order equation is quite different from the classical Laplace equation, however, we can deal with the semigroup $e^{-t\sqrt{L}}$ by subordination formula, and then we can characterize the molecular Hardy spaces $H_L^1(\mathbb{R}^n)$ by the area integral $\mathcal{S}_P^{L,k}$ ($k \in \mathbb{N}$) defined by $e^{-t\sqrt{L}}$. Let us begin with some definitions and lemmas.

Let L be the homogeneous elliptic operator of $2m$ order in divergence form defined by (1.4)–(1.6). For $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$, the area integral $\mathcal{S}_P^{L,k}$ associated with the semigroup $e^{-t\sqrt{L}}$ is defined by

$$\mathcal{S}_P^{L,k} f(x) = \left(\iint_{\Gamma(x)} |(t^m \sqrt{L})^k e^{-t\sqrt{L}} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}. \quad (6.1)$$

We denote $\mathcal{S}_P^{L,1}$ by \mathcal{S}_P^L for simplicity. The purpose of this section is to show that the Hardy space H_L^1 can be characterized by the area integral $\mathcal{S}_P^{L,k}$. First we introduce a Hardy space $H_{\mathcal{S}_P^{L,k}}^1(\mathbb{R}^n)$ as follows.

Definition 6.1 (Hardy space $H_{\mathcal{S}_P^{L,k}}^1$). Letting $k \in \mathbb{N}$, the space $H_{\mathcal{S}_P^{L,k}}^1$ is defined as the completion of the space

$$\{f \in L^2: \mathcal{S}_P^{L,k} f \in L^1(\mathbb{R}^n)\}$$

with respect to the norm $\|f\|_{H_{\mathcal{S}_P^{L,k}}^1} := \|\mathcal{S}_P^{L,k} f\|_{L^1}$.

Now we state our main result in this section:

Theorem 6.1.

(a) For every $(2, \varepsilon, M)$ -representation $f = \sum_{j=0}^{\infty} \lambda_j u_j$, the series converges in $H_{\mathcal{S}_P^{L,k}}^1$ for all $k \in \mathbb{N}$ and

$$\left\| \sum_{j=0}^{\infty} \lambda_j u_j \right\|_{H_{\mathcal{S}_P^{L,k}}^1(\mathbb{R}^n)} \leq C \sum_{j=0}^{\infty} |\lambda_j|.$$

In particular, we have $\|f\|_{H_{\mathcal{S}_P^{L,k}}^1(\mathbb{R}^n)} \leq C \|f\|_{H_L^1(\mathbb{R}^n)}$.

(b) Suppose $k \in \mathbb{N}$, $M > \frac{n}{4m}(2m-1)$ and $0 < \varepsilon < (2M+2k+1)\frac{m}{2m-1} - \frac{n}{2}$. If $f \in L^2(\mathbb{R}^n)$ with $\|\mathcal{S}_P^{L,k} f\|_{L^1(\mathbb{R}^n)} < \infty$, then $f \in \tilde{H}_L^1(\mathbb{R}^n)$. Furthermore, there exist a sequence of $(2, \varepsilon, M)$ -molecules $\{u_j\}_{j=0}^\infty$ and $\{\lambda_j\}_{j=0}^\infty$ such that $f = \sum_{j=0}^\infty \lambda_j u_j$, with

$$\|f\|_{\tilde{H}_L^1(\mathbb{R}^n)} \leq C \sum_{j=0}^\infty |\lambda_j| \leq C \|\mathcal{S}_P^{L,k} f\|_{L^1(\mathbb{R}^n)}. \quad (6.2)$$

The proof of Theorem 6.1 will be given in Sections 6.2 and 6.3, respectively. As an immediate consequence of Theorem 6.1, we have

Corollary 6.1. $H_L^1(\mathbb{R}^n) = H_{S_P^{L,k}}^1(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ with $\|f\|_{H_L^1(\mathbb{R}^n)} \approx \|f\|_{H_{S_P^{L,k}}^1(\mathbb{R}^n)}$.

6.1. Square functions associated to semigroup $e^{-t\sqrt{L}}$

Let us begin with giving the definitions and some important properties of the square functions associated to semigroup $e^{-t\sqrt{L}}$. First we consider the definition of the vertical square function $g_P^{L,k}$. For $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$, then $g_P^{L,k}$ is defined by

$$g_P^{L,k} f(x) = \left(\int_0^\infty |(t\sqrt{L})^k e^{-t\sqrt{L}} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Similarly, we denote by g_P^L for $g_P^{L,1}$. Below we will show that $g_P^{L,k}$ and $\mathcal{S}_P^{L,k}$ share the same properties as $g_h^{L,k}$ and $\mathcal{S}_h^{L,k}$ (see Section 3).

First we show that $g_P^{L,k} f$ can be dominated by $g_h^{L,k'} f$ pointwise.

Lemma 6.1. Let $f \in C_0^\infty(\mathbb{R}^n)$. Then

- (i) $g_P^{L,2\ell} f(x) \leq C_\ell g_h^{L,\ell} f(x)$ for $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^n$.
- (ii) $g_P^{L,2\ell+1} f(x) \leq C_\ell g_h^{L,\ell+1} f(x)$ for $\ell \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}^n$.

Proof. (i) By the subordination formula

$$e^{-t\sqrt{L}} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} e^{-\frac{t^2 L}{4z}} f dz, \quad (6.3)$$

Minkowski's inequality and making the change of variables $s^2 := \frac{t^2}{4z}$, we have that

$$g_P^{L,2\ell} f(x) = \left(\int_0^\infty |(t\sqrt{L})^{2\ell} e^{-t\sqrt{L}} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\begin{aligned}
&= C \left(\int_0^\infty \left| \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (t^2 L)^\ell e^{-\frac{t^2 L}{4z}} f(x) dz \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\leq C_\ell g_h^{L,\ell} f(x).
\end{aligned}$$

(ii) Observe that $t\sqrt{L}e^{-t\sqrt{L}} = -t\partial_t e^{-t\sqrt{L}}$. Then using (6.3) again and making the change of variables $s^2 := \frac{t^2}{4z}$, we get

$$\begin{aligned}
g_P^{L,2\ell+1} f(x) &= \left(\int_0^\infty |(t\sqrt{L})^{2\ell+1} e^{-t\sqrt{L}} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&= C \left(\int_0^\infty \left| \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (t^2 L)^\ell t \partial_t e^{-\frac{t^2 L}{4z}} f(x) dz \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^\infty \left| \int_0^\infty e^{-z} z^{-\frac{3}{2}} (t^2 L)^{\ell+1} e^{-\frac{t^2 L}{4z}} f(x) dz \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\leq C_\ell g_h^{L,\ell+1} f(x). \quad \square
\end{aligned}$$

Below we give two consequences of Lemma 6.1.

Lemma 6.2. For $k \in \mathbb{N}$ and all closed sets E, F in \mathbb{R}^n with $d(E, F) > 0$, the following results hold:

(i) If $p_L < p \leq 2$ and $f \in L^p(\mathbb{R}^n)$ supported in E , then

$$\begin{aligned}
&\|g_P^{L,k}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} + \|g_P^{L,k}(tLe^{-tL})^M f\|_{L^2(F)} \\
&\leq Ct^{\frac{1}{2m}(\frac{n}{2} - \frac{n}{p})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}.
\end{aligned}$$

(ii) If $2 \leq p < \tilde{p}_L$ and $f \in L^2(\mathbb{R}^n)$ supported in E , then

$$\begin{aligned}
&\|g_P^{L,k}(\mathbf{I} - e^{-tL})^M f\|_{L^p(F)} + \|g_P^{L,k}(tLe^{-tL})^M f\|_{L^p(F)} \\
&\leq Ct^{\frac{1}{2m}(\frac{n}{p} - \frac{n}{2})} \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}.
\end{aligned}$$

Proof. The conclusions are obvious by Lemmas 6.1 and 3.3. \square

Theorem 6.2. $g_P^{L,k}$ is bounded on $L^p(\mathbb{R}^n)$ for $p_L < p < \tilde{p}_L$ and $k \in \mathbb{N}$.

Proof. This is a direct result of Lemma 6.1 and Theorem 3.3. \square

Now we show that $\{S_P^{L,k}\}_{k \in \mathbb{N}}$ are all bounded on $L^p(\mathbb{R}^n)$ for $p_L < p < \tilde{p}_L$.

Theorem 6.3. For $p_L < p < \tilde{p}_L$ and $k \in \mathbb{N}$, $S_P^{L,k}$ is a bounded operator on $L^p(\mathbb{R}^n)$.

Proof. The proof of Theorem 6.3 is very similar to Theorem 3.4, so we only give an outline of the proof here. For $k \geq 1$, using (3.28) and the L^2 boundedness of $g_P^{L,k}$ (Theorem 6.2), we get

$$\|S_P^{L,k} f\|_{L^2} \leq C \left(\iint_{\mathbb{R}_+^{n+1}} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}} f(y)|^2 \frac{dy dt}{t} \right)^{\frac{1}{2}} \leq C \|g_P^{L,k} f\|_{L^2} \leq C \|f\|_{L^2}.$$

(i) *Case where $p_L < p < 2$.* As done before, we will apply Theorem A in this case. Denote $A_r = \mathbf{I} - (\mathbf{I} - e^{-r^{2m}L})^M$, where $M > \frac{n}{4m}(2m-1)$ and $r = r(B)$ is the radius of ball B . By (3.25), we only need to verify that (3.21) holds for $S_P^{L,k}$ and p_0 , where p_0 satisfies $p_L < p_0 < p < 2$.

Notice that $S_j(B)$ ($j \geq 0$) are defined as in Section 4, for $j \geq 3$ and f supported in B , by (3.28) we obtain

$$\begin{aligned} & \int_{S_j(B)} |S_P^{L,k}(\mathbf{I} - A_{r(B)})f(x)|^2 dx \\ & \leq C \int_{\mathcal{R}(S_j(B))} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}}(\mathbf{I} - e^{-r^{2m}L})^M f(x)|^2 \frac{dx dt}{t} \\ & \leq C \int_{\mathbb{R}^n \setminus 2^{j-1}B} \int_0^\infty |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}}(\mathbf{I} - e^{-r^{2m}L})^M f(x)|^2 \frac{dt}{t} dx \\ & \quad + C \sum_{\ell=0}^{j-2} \int_{S_\ell(B)} \int_{(2^{j-1}-2^\ell)r}^\infty |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}}(\mathbf{I} - e^{-r^{2m}L})^M f(x)|^2 \frac{dt}{t} dx \\ & := I_P + \sum_{\ell=0}^{j-2} I_{P,\ell}. \end{aligned} \tag{6.4}$$

For I_P , similarly to (3.30), by the conclusion (i) of Lemma 6.2 we have

$$I_P = C \|g_P^{L,k}(\mathbf{I} - e^{-r^{2m}L})^M f\|_{L^2(\mathbb{R}^n \setminus 2^{j-1}B)}^2 \leq C r^{-2n(\frac{1}{p_0}-\frac{1}{2})} 2^{-\frac{4mjM}{2m-1}} \|f\|_{L^{p_0}(B)}^2. \tag{6.5}$$

To estimate $I_{P,\ell}$, by the subordination formula (6.3) we get

$$e^{-t^m \sqrt{L}} f = C \int_0^\infty \frac{e^{-z}}{\sqrt{z}} e^{-\frac{t^{2m}L}{4z}} f dz. \tag{6.6}$$

By (6.6) and the fact $-mt^m\sqrt{L}e^{-t^m\sqrt{L}} = t\partial_t e^{-t^m\sqrt{L}}$, we have

$$\begin{aligned} & \left(\int_{(2^{j-1}-2^\ell)_r}^\infty |(t^m\sqrt{L})^k e^{-t^m\sqrt{L}}(\mathbf{I} - e^{r^{2m}L})^M f(x)|^2 \frac{dt}{t} \right)^{1/2} \\ & \leq \begin{cases} C(\int_{(2^{j-1}-2^\ell)_r}^\infty |(t^{2m}L)^v e^{-\frac{t^{2m}L}{4z}}(\mathbf{I} - e^{-r^{2m}L})^M f(x) dz|^2 \frac{dt}{t})^{1/2}, \\ \text{if } k = 2v, \\ C(\int_{(2^{j-1}-2^\ell)_r}^\infty |(t^{2m}L)^{v+1} e^{-\frac{t^{2m}L}{4z}}(\mathbf{I} - e^{-r^{2m}L})^M f(x) dz|^2 \frac{dt}{t})^{1/2}, \\ \text{if } k = 2v + 1, \end{cases} \\ & \leq \begin{cases} C_{M,v} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (\int_{\frac{(2^j r)^{2m}}{cz}}^\infty |(szL)^v e^{-sv(M+1)L}(\mathbf{I} - e^{-r^{2m}L})^M f(x)|^2 \frac{ds}{s})^{1/2} dz, \\ \text{if } k = 2v, \\ C_{M,v} \int_0^\infty \frac{e^{-z}}{z^{3/2}} (\int_{\frac{(2^j r)^{2m}}{cz}}^\infty |(szL)^{v+1} e^{-s(v+1)(M+1)L}(\mathbf{I} - e^{-r^{2m}L})^M f(x)|^2 \frac{ds}{s})^{1/2} dz, \\ \text{if } k = 2v + 1, \end{cases} \end{aligned}$$

where we use the following transforms: $\frac{t^{2m}}{4z} = sv(M+1)$ for $k = 2v$, and $\frac{t^{2m}}{4z} = s(v+1)(M+1)$ for $k = 2v + 1$, respectively. Hence, if $k = 2v$, we have

$$\begin{aligned} I_{P,\ell}^{1/2} & \leq C_{M,v} \int_0^\infty \frac{z^v e^{-z}}{\sqrt{z}} \left(\int_{\frac{(2^j r)^{2m}}{cz}}^\infty \|(sL)^v e^{-sv(M+1)L}(\mathbf{I} - e^{-r^{2m}L})^M f(x)\|_{L^2(S_\ell(B))}^2 \frac{ds}{s} \right)^{1/2} dz \\ & \leq C_{M,v} \int_0^\infty \frac{z^v e^{-z}}{\sqrt{z}} \left(\int_{\frac{(2^j r)^{2m}}{cz}}^\infty \left(\frac{r^{2m}}{sv} \right)^{2M} \right. \\ & \quad \times \left\| \left(\frac{sv}{r^{2m}} (e^{-svL} - e^{-(r^{2m}+sv)L}) \right)^M (sLe^{-sL})^v f(x) \right\|_{L^2(S_\ell(B))}^2 \frac{ds}{s} \right)^{1/2} dz. \end{aligned}$$

Similarly to (3.31), applying the conclusions (iii) and (iv) in Theorem 3.2, then by Lemma 3.2 we get

$$\begin{aligned} I_{P,\ell}^{1/2} & \leq C_{M,v} \int_0^\infty \frac{z^v e^{-z}}{\sqrt{z}} \left(\int_{\frac{(2^j r)^{2m}}{cz}}^\infty \left(\frac{r^{2m}}{sv} \right)^{2M} (sv)^{\frac{1}{m}(\frac{n}{2} - \frac{n}{p_0})} e^{-(\frac{d(S_\ell(B),B)}{c(sv)^{1/2m})} \frac{2m}{2m-1}} \frac{ds}{s} \right)^{1/2} dz \|f\|_{L^{p_0}(B)} \\ & \leq C'_{M,v} r^{2mM} \int_0^\infty \frac{z^v e^{-z}}{\sqrt{z}} \left(\int_{\frac{(2^j r)^{2m}}{cz}}^\infty s^{\frac{n}{m}(\frac{1}{2} - \frac{1}{p_0})} \frac{ds}{s^{2M+1}} \right)^{1/2} dz \|f\|_{L^{p_0}(B)} \\ & \leq C'_{M,v} r^{-n(\frac{1}{p_0} - \frac{1}{2})} 2^{-2mjM} \|f\|_{L^{p_0}(B)}. \end{aligned}$$

For $k = 2\nu + 1$, using the same idea as above, we have

$$\begin{aligned} I_{P,\ell}^{1/2} &\leq C_{M,\nu} \int_0^\infty \frac{z^\nu e^{-z}}{\sqrt{z}} \left(\int_{\frac{(2j_r)^{2m}}{cz}}^\infty \left(\frac{r^{2m}}{s} \right)^{2M} \right. \\ &\quad \times \left. \left\| \left(\frac{s}{r^{2m}} (e^{-s(\nu+1)L} - e^{-(r^{2m}+s(\nu+1))L}) \right)^M (sLe^{-sL})^{(\nu+1)} f(x) \right\|_{L^2(S_\ell(B))}^2 \frac{ds}{s} \right)^{1/2} dz \\ &\leq C'_{M,\nu} r^{-n(\frac{1}{p_0}-\frac{1}{2})} 2^{-2mjM} \|f\|_{L^{p_0}(B)}. \end{aligned}$$

Thus we show that for any $k \in \mathbb{N}$,

$$\left(\frac{1}{|2^{j+1}B|} \int_{S_j(B)} |S_P^{L,k}(\mathbf{I} - A_{r(B)})f(x)|^2 dx \right)^{\frac{1}{2}} \leq g(j) \left(\frac{1}{|B|} \int_B |f(x)|^{p_0} dx \right)^{\frac{1}{p_0}},$$

where $g(j) = C_{M,k} 2^{-j(\frac{n}{2} + \frac{2mM}{2m-1})}$. So, (3.21) holds for $S_P^{L,k}$. Hence, $S_P^{L,k}$ is bounded in $L^p(\mathbb{R}^n)$ for $p_L < p \leq 2$.

(ii) *Case where $2 < p < \tilde{p}_L$.* Similarly to the proof of Theorem 3.4, we can easily obtain the L^p boundedness of $S_P^{L,k}$ for all $2 < p < \tilde{p}_L$ by using Lemma 3.5 and Theorem 6.2. We omit the details here. \square

Finally, we give a useful estimate on the family of operators $\{(t^m \sqrt{L})^k e^{-t^m \sqrt{L}}\}_{k \in \mathbb{N}}$.

Lemma 6.3. *For all closed sets E, F in \mathbb{R}^n with $d(E, F) > t > 0$ and $f \in L^2(\mathbb{R}^n)$ supported in E*

$$\|(t^m \sqrt{L})^{2\nu} e^{-t^m \sqrt{L}} f\|_{L^2(F)} \leq C_\nu \left(\frac{t}{d(E, F)} \right)^{(2\nu+1)\frac{m}{2m-1}} \|f\|_{L^2(E)}, \quad \forall \nu = 1, 2, \dots \quad (6.7)$$

and

$$\|(t^m \sqrt{L})^{2\nu+1} e^{-t^m \sqrt{L}} f\|_{L^2(F)} \leq C_\nu \left(\frac{t}{d(E, F)} \right)^{(2\nu+1)\frac{m}{2m-1}} \|f\|_{L^2(E)}, \quad \forall \nu = 0, 1, \dots \quad (6.8)$$

Proof. By the subordination formula (6.6) and Minkowski's inequality, and making the change of variables $s = (\frac{zd(E,F)^{2m}}{t^{2m}})^{1/(2m-1)}$ we have

$$\begin{aligned} \|(t^m \sqrt{L})^{2\nu} e^{-t^m \sqrt{L}} f\|_{L^2(F)} &\leq C \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \left\| \left(\frac{t^{2m} L}{4z} \right)^\nu e^{-\frac{t^{2m} L}{4z}} f \right\|_{L^2(F)} z^\nu dz \\ &\leq C \int_0^\infty e^{-z} z^{\nu-\frac{1}{2}} e^{-\left(\frac{zd(E,F)^{2m}}{ct^{2m}}\right)\frac{1}{2m-1}} dz \|f\|_{L^2(E)} \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^\infty e^{-s} \left(\frac{s^{2m-1} t^{2m}}{d(E, F)^{2m}} \right)^{v-\frac{1}{2}} e^{-\frac{s^{2m-1} t^{2m}}{d(E, F)^{2m}}} \frac{s^{2m-2} t^{2m}}{d(E, F)^{2m}} ds \|f\|_{L^2(E)} \\
&\leq C \left(\frac{t}{d(E, F)} \right)^{(2v+1)m} \int_0^\infty e^{-s} s^{(2m-1)(v-\frac{1}{2})+2m-2} ds \|f\|_{L^2(E)} \\
&\leq C \left(\frac{t}{d(E, F)} \right)^{(2v+1)m} \|f\|_{L^2(E)} \\
&\leq C \left(\frac{t}{d(E, F)} \right)^{(2v+1)m/(2m-1)} \|f\|_{L^2(E)}.
\end{aligned}$$

On the other hand, observe that $-mt^m \sqrt{L} e^{-t^m \sqrt{L}} = t \partial_t e^{-t^m \sqrt{L}}$. Then by (6.6) and using Lemmas 3.1 and 3.2, we get

$$\begin{aligned}
\|(t^m \sqrt{L})^{2v+1} e^{-t^m \sqrt{L}} f\|_{L^2(F)} &\leq C \left\| \int_0^\infty \frac{e^{-z}}{\sqrt{z}} (t^{2m} L)^v t \partial_t e^{-\frac{t^{2m} L}{4z}} f dz \right\|_{L^2(F)} \\
&\leq C \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \left\| \left(\frac{t^{2m} L}{4z} \right)^{v+1} e^{-\frac{t^{2m} L}{4z}} f \right\|_{L^2(F)} z^v dz \\
&\leq C \int_0^\infty e^{-z} z^{v-\frac{1}{2}} e^{-\left(\frac{z d(E, F)^{2m}}{c t^{2m}}\right)^{\frac{1}{2m-1}}} dz \|f\|_{L^2(E)} \\
&\leq C_k \left(\frac{t}{d(E, F)} \right)^{(2v+1)\frac{m}{2m-1}} \|f\|_{L^2(E)}.
\end{aligned}$$

Thus, we complete the proof of lemma. \square

6.2. Proof of the conclusion (a) of Theorem 6.1

Unlike the proof of Theorem 5.1, one can see from the previous subsection that the proof here would depend on the parity of k . By Theorem 6.3 and Lemma 5.1, we only need to prove that

$$\|\mathcal{S}_P^{L,k} u\|_{L^1(\mathbb{R}^n)} \leq C$$

for every $(2, \varepsilon, M)$ -molecules u (see Definition 4.1). To do this, we write

$$\|\mathcal{S}_P^{L,k} u\|_{L^1(\mathbb{R}^n)} \leq \|\mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)^{2m} L})^M u\|_{L^1(\mathbb{R}^n)} + \|\mathcal{S}_P^{L,k} (\mathbf{I} - (\mathbf{I} - e^{-l(Q)^{2m} L})^M) u\|_{L^1(\mathbb{R}^n)},$$

where Q denotes the cube related to molecules u and $l(Q)$ is its side length. Like (5.6), we have

$$\begin{aligned} & \| \mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)2^m L})^M u \|_{L^1(\mathbb{R}^n)} \\ & \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (2^{i+j} l(Q))^{\frac{n}{2}} \| \mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u) \|_{L^2(S_j(Q_i))}. \end{aligned}$$

Note that $\mathcal{S}_P^{L,k}$ and $(\mathbf{I} - e^{-l(Q)2^m L})^M$ are both bounded on $L^2(\mathbb{R}^n)$, thus for $j = 0, 1, 2$, we get

$$\| \mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u) \|_{L^2(S_j(Q_i))} \leq C \| u \|_{L^2(S_i(Q))}. \quad (6.9)$$

Now let us turn to the case $j \geq 3$. By (3.28) in Lemma 3.4, like (6.4) we get

$$\begin{aligned} & \| \mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u) \|_{L^2(S_j(Q_i))}^2 \\ & \leq C \iint_{\mathcal{R}(S_j(Q_i))} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u)(x)|^2 \frac{dx dt}{t} \\ & \leq C \int_{\mathbb{R}^n \setminus Q_{i+j-2}} \int_0^{\infty} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u)(x)|^2 \frac{dt}{t} dx \\ & \quad + C \sum_{\ell=0}^{j-2} \int_{S_{\ell}(Q_i)} \int_{(2^{j-1}-2^{\ell})2^i l(Q)}^{\infty} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u)(x)|^2 \frac{dt}{t} dx \\ & := I + \sum_{\ell=0}^{j-2} I_{\ell}. \end{aligned}$$

Similarly to (6.4), using the conclusion (i) of Lemma 6.2 we have

$$\begin{aligned} I & = C \| g_P^{L,k} (\mathbf{I} - e^{-r^{2m} L})^M (\chi_{S_i(Q)} u) \|_{L^2(\mathbb{R}^n \setminus Q_{i+j-2})}^2 \\ & \leq C (2^{i+j})^{-\frac{4mM}{2m-1}} \| u \|_{L^2(S_i(Q))}^2. \end{aligned} \quad (6.10)$$

Concerning I_{ℓ} , $\ell = 0, 1, \dots, j-2$, applying the same method of estimating $I_{P,\ell}$ in Theorem 6.3 (for the case $k = 1$), we may get

$$I_{\ell} \leq C (2^{i+j})^{-4mM} \| u \|_{L^2(S_i(Q))}^2. \quad (6.11)$$

Thus, by (6.9)–(6.11) we have

$$\begin{aligned} & \| \mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)2^m L})^M u \|_{L^1(\mathbb{R}^n)} \\ & \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (2^{i+j} l(Q))^{\frac{n}{2}} \| \mathcal{S}_P^{L,k} (\mathbf{I} - e^{-l(Q)2^m L})^M (\chi_{S_i(Q)} u) \|_{L^2(S_j(Q_i))} \end{aligned}$$

$$\leq C \sum_{i=0}^{\infty} \left\{ \sum_{j=3}^{\infty} \left[2^{-(i+j)(\frac{2mM}{2m-1})} + \sqrt{j} 2^{-(i+j)(2mM)} \right] (2^{i+j} l(Q))^{\frac{n}{2}} + (2^i l(Q))^{\frac{n}{2}} \right\} \|u\|_{L^2(S_i(Q))} \\ \leq C.$$

Finally, using a similar argument of proving (5.5) we may show that

$$\|S_P^{L,k} (\mathbf{I} - (\mathbf{I} - e^{-l(Q)^{2m}L})^M) u\|_{L^1(\mathbb{R}^n)} \leq C.$$

We therefore finish the proof of (a) in Theorem 6.1.

6.3. Proof of the conclusion (b) of Theorem 6.1

The proof of this theorem is quite similar to the argument used in the proof of the conclusion (b) of Theorem 5.1. We just need to make minor modifications and write down the main steps. To be more precise, since L is an m -accretive operator, it follows from Kato [47, p. 281] that L has a unique m -accretive square root $L^{\frac{1}{2}}$ such that $L^{\frac{1}{2}} L^{\frac{1}{2}} = L$, which means that the $L^2(\mathbb{R}^n)$ functional calculus still holds for $L^{\frac{1}{2}}$. Thus we begin the proof with giving the following Calderón reproducing formula (in the sense of $L^2(\mathbb{R}^n)$):

$$f = C_{M,m,k} \int_0^{\infty} ((t^m \sqrt{L})^{2M+2k} e^{-t^m \sqrt{L}})^2 f \frac{dt}{t}.$$

Now, let

$$O_\ell := \{x \in \mathbb{R}^n : S_P^{L,k} f(x) > 2^\ell\}$$

and denote

$$u_\ell^j(N) = \frac{1}{\lambda_\ell^j} \int_{\frac{1}{N}}^N (t^m \sqrt{L})^{4M+3k} e^{-t^m \sqrt{L}} (\chi_{T_\ell^j} (t^m \sqrt{L})^k e^{-t^m \sqrt{L}}) f \frac{dt}{t},$$

where T_ℓ^j ($\ell, j \in \mathbb{Z}$) is defined as (5.11). Choosing $h \in L^2(S_i(Q_\ell^j))$ such that $\|h\|_{L^2(S_i(Q_\ell^j))} = 1$, and setting $\chi_\ell^j := \chi_{(Q_\ell^j \times (0, \infty)) \cap \widehat{O}_\ell^*}$, we have

$$\left| \int_{\mathbb{R}^n} u_\ell^j(N)(x) \overline{h(x)} dx \right| \\ \leq \frac{C}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap Q_\ell^j} \iint_{\Gamma(x)} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}} f(y)|^2 \frac{dy dt}{t^{n+1}} dx \right)^{\frac{1}{2}}$$

$$\times \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \iint_{\Gamma(x) \cap (Q_\ell^j \times (0, \infty)) \cap \widehat{O}_\ell^*} |(t^m \sqrt{L})^{4M+3k} e^{-t^m \sqrt{L}})^* h(y)|^2 \frac{dy dt}{t^{n+1}} dx \right)^{\frac{1}{2}} \\ := I_1 \times I_2.$$

For I_1 we have

$$I_1 \leq \frac{C}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} (S_P^{L,k} f(x))^2 dx \right)^{\frac{1}{2}} \leq C |Q_\ell^j|^{-\frac{1}{2}}. \quad (6.12)$$

To deal with I_2 , when $i \leq 4$, using the L^p boundedness of $S_P^{L,4M+3k}$ (see Theorem 6.3), we have

$$I_2 \leq C \|S_P^{L,4M+3k} h\|_{L^2(\mathbb{R}^n)} \leq C \|h\|_{L^2(S_i(Q_\ell^j))} \leq C.$$

When $i \geq 5$, we involve the same method used in (5.15) and the estimates (6.7) and (6.8) to obtain

$$I_2 \leq C \left(\int_{3CQ_k^j} \int_0^{cl(Q_\ell^j)} |(t^m \sqrt{L})^{4M+3k} e^{-t^m \sqrt{L}})^* h(y)|^2 \frac{dt}{t} dy \right)^{\frac{1}{2}} \\ \leq C \left(\int_0^{cl(Q_\ell^j)} \|((t^m \sqrt{L})^{4M+3k} e^{-t^m \sqrt{L}})^* h\|_{L^2(3CQ_\ell^j)}^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\ \leq \begin{cases} C 2^{-i(4M+3k+1)\frac{m}{2m-1}}, & k \text{ is even,} \\ C 2^{-i(4M+3k)\frac{m}{2m-1}}, & k \text{ is odd,} \end{cases} \quad (6.13)$$

which allows to get that

$$\|u_\ell^j(N)\|_{L^2(S_i(Q_\ell^j))} \leq \begin{cases} C 2^{-i(4M+3k+1)\frac{m}{2m-1}} |Q_\ell^j|^{-\frac{1}{2}}, & k \text{ is even,} \\ C 2^{-i(4M+3k)\frac{m}{2m-1}} |Q_\ell^j|^{-\frac{1}{2}}, & k \text{ is odd.} \end{cases} \quad (6.14)$$

Now turning to compute $\|(l(Q_\ell^j)^{2m} L)^{-M} u_\ell^j(N)\|_{L^2(S_i(Q_\ell^j))}$. Similarly, we write

$$\left| \int_{\mathbb{R}^n} (l(Q_\ell^j)^{2m} L)^{-M} u_\ell^j(N)(x) \overline{h(x)} dx \right| \\ \leq \frac{C}{\lambda_\ell^j} \left(\int_{O_{\ell+1}^c \cap cQ_\ell^j} \iint_{\Gamma(x)} |(t^m \sqrt{L})^k e^{-t^m \sqrt{L}} f(y)|^2 \frac{dy dt}{t^{n+1}} dx \right)^{\frac{1}{2}}$$

$$\begin{aligned}
& \times \left(\int_{O_{\ell+1}^c \cap c Q_\ell^j} \iint_{\Gamma(x) \cap (Q_\ell^j \times (0, \infty)) \cap \widehat{O}_\ell^*} \left| \left(\frac{t}{l(Q_\ell^j)} \right)^{2mM} \right. \right. \\
& \times \left. \left. \left((t^m \sqrt{L})^{2M+3k} e^{-t^m \sqrt{L}} \right)^* h(y) \right|^2 \frac{dy dt}{t^{n+1}} dx \right)^{\frac{1}{2}} \\
& := \tilde{I}_1 \times \tilde{I}_2.
\end{aligned}$$

Notice that (6.12) holds for \tilde{I}_1 since $\tilde{I}_1 = I_1$. As for \tilde{I}_2 , the same argument with (6.13) can lead to

$$\tilde{I}_2 \leq \begin{cases} C 2^{-i(2M+3k+1)\frac{m}{2m-1}}, & k \text{ is even,} \\ C 2^{-i(2M+3k)\frac{m}{2m-1}}, & k \text{ is odd.} \end{cases}$$

Then we have

$$\| (l(Q_\ell^j)^{2m} L)^{-M} u_\ell^j(N) \|_{L^2(S_i(Q_\ell^j))} \leq \begin{cases} C 2^{-i(2M+3k+1)\frac{m}{2m-1}} |Q_\ell^j|^{-\frac{1}{2}}, & k \text{ is even,} \\ C 2^{-i(2M+3k)\frac{m}{2m-1}} |Q_\ell^j|^{-\frac{1}{2}}, & k \text{ is odd.} \end{cases} \quad (6.15)$$

Therefore by assumptions $k \in \mathbb{N}$, $M > \frac{n}{4m}(2m-1)$ and $0 < \varepsilon < (2M+3k)\frac{m}{2m-1} - \frac{n}{2}$, combining (6.14) with (6.15) we have

$$\sum_{i=0}^{\infty} 2^{i(\frac{n}{2}+\varepsilon)} |Q_\ell^j|^{\frac{1}{2}} \sum_{k=0}^M \| (l(Q)^{-2m} L^{-1})^\ell u_k^j(N) \|_{L^p(S_i(Q))} \leq C. \quad (6.16)$$

To be more rigorous, we need to prove $f_N \rightarrow f$ in $\tilde{H}_L^1(\mathbb{R}^n)$ and $\|\mathcal{S}_p^{L,k}(f_N - f)\|_{L^1} \rightarrow 0$. As the details are similar to the proof of (b) of Theorem 5.1, we omit the details here. Thus we finish the proof of the conclusion (b) of Theorem 6.1.

7. (H_L^1, L^1) boundedness of some operators

In this section, as some applications, we give the (H_L^1, L^1) boundedness of some operators, where and in the sequel, L still denotes the homogeneous elliptic operator of order $2m$ in divergence form defined by (1.4)–(1.6). First we show a general conclusion as follows.

Theorem 7.1. *Let $p_L < p \leq 2$ and assume that the sublinear operator*

$$T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$$

satisfies the following estimate: There exist $M \in \mathbb{N}$, $M > \frac{n}{4m}(2m-1)$, such that for all closed sets E, F with $d(E, F) > 0$ and every $f \in L^p(\mathbb{R}^n)$ supposed in E

$$\| T(\mathbf{I} - e^{-tL})^M f \|_{L^p(F)} \leq C \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \| f \|_{L^p(E)} \quad (7.1)$$

and

$$\|T(tLe^{-tL})^M f\|_{L^p(F)} \leq C \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^p(E)}. \quad (7.2)$$

Then T can be extended to a bounded operator from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

Proof. By Lemma 5.1, we only need to prove that (5.3) holds for any (p, ε, M) -molecule u . To begin, we write

$$\|Tu\|_{L^1(\mathbb{R}^n)} \leq \|T(\mathbf{I} - e^{-l(Q)^{2m}L})^M u\|_{L^1(\mathbb{R}^n)} + \|T(\mathbf{I} - (\mathbf{I} - e^{-l(Q)^{2m}L})^M)u\|_{L^1(\mathbb{R}^n)} := I_1 + I_2,$$

and for I_1 , we can write

$$I_1 \leq \sum_{i=0}^{\infty} \|T(\mathbf{I} - e^{-l(Q)^{2m}L})^M (\chi_{S_i(Q)} u)\|_{L^1(\mathbb{R}^n)},$$

where the family of annuli $\{S_i(Q)\}_{i=0}^{\infty}$ is taken with respect to the cube Q associated with u . Hence,

$$\begin{aligned} & \|T(\mathbf{I} - e^{-l(Q)^{2m}L})^M (\chi_{S_i(Q)} u)\|_{L^1(\mathbb{R}^n)} \\ & \leq C \sum_{j=0}^{\infty} (2^{i+j}l(Q))^{n-\frac{n}{p}} \|T(\mathbf{I} - e^{-l(Q)^{2m}L})^M (\chi_{S_j(Q_i)} u)\|_{L^p(S_j(Q_i))} \\ & \leq C \sum_{j=2}^{\infty} (2^{i+j}l(Q))^{n-\frac{n}{p}} \left(\frac{l(Q)^{2m}}{d(S_j(Q_i), S_i(Q))^{2m}} \right)^{\frac{M}{2m-1}} \|u\|_{L^p(S_i(Q))} \\ & \quad + C(2^i l(Q))^{n-\frac{n}{p}} \|u\|_{L^p(S_i(Q))} \\ & \leq C(2^i l(Q))^{n-\frac{n}{p}} \|u\|_{L^p(S_i(Q))} \sum_{j=2}^{\infty} 2^{j(n-\frac{n}{p}-2mM/(2m-1))}, \end{aligned}$$

where we use (7.1) and the uniform boundedness of the family of operators $\{e^{-tL}\}_{t>0}$ in $L^p(\mathbb{R}^n)$. Then we have

$$I_1 \leq \sum_{i=0}^{\infty} \|T(\mathbf{I} - e^{-l(Q)^{2m}L})^M (\chi_{S_i(Q)} u)\|_{L^1(\mathbb{R}^n)} \leq C \sum_{i=0}^{\infty} (2^i l(Q))^{n-\frac{n}{p}} \|u\|_{L^p(S_i(Q))}. \quad (7.3)$$

As for I_2 , we observe that

$$\mathbf{I} - (\mathbf{I} - e^{-l(Q)^{2m}L})^M = \sum_{k=1}^M \binom{M}{k} (-1)^{k+1} e^{-kl(Q)^{2m}L},$$

where $\binom{M}{k} = \frac{M!}{(M-k)!k!}$, $k = 1, 2, \dots, M$. Therefore

$$\|T(\mathbf{I} - (\mathbf{I} - e^{-l(Q)^{2m}L})^M)u\|_{L^1(\mathbb{R}^n)} \leq C \sup_{1 \leq k \leq M} \|Te^{-kl(Q)^{2m}L}u\|_{L^1(\mathbb{R}^n)}.$$

Since

$$\|Te^{-kl(Q)^{2m}L}u\|_{L^1(\mathbb{R}^n)} = C \left\| T \left(\frac{k}{M} l(Q)^{2m} L e^{-\frac{k}{M} l(Q)^{2m} L} \right)^M (l(Q)^{-2m} L^{-1})^M u \right\|_{L^1(\mathbb{R}^n)},$$

we use a method similar to the one used above to estimate I_2 as long as replacing u and $(\mathbf{I} - e^{-l(Q)^{2m}L})^M$ by $(l(Q)^{-2m} L^{-1})^M u$ and $(\frac{k}{M} l(Q)^{2m} L e^{-\frac{k}{M} l(Q)^{2m} L})^M$, respectively. Thus, by (7.2) we may get

$$I_2 \leq C \sum_{i=0}^{\infty} (2^i l(Q))^{n-\frac{n}{p}} \|(l(Q)^{-2m} L^{-1})^M u\|_{L^p(S_i(Q))}. \quad (7.4)$$

Hence, by (7.3), (7.4) and Definition 4.1 of (p, ε, M) -molecule, we finish the proof of Theorem 7.1. \square

7.1. Riesz transforms $\nabla^m L^{-1/2}$

It is well known that the Riesz transform has a basic importance in harmonic analysis and PDE. Denote by $\nabla^m L^{-1/2}$ the Riesz transforms associated with the homogeneous elliptic operator L of order $2m$ in divergence form defined by (1.4)–(1.6). In [2], Auscher gave the necessary and sufficient conditions for the L^p -boundedness of Riesz transforms $\nabla^m L^{-1/2}$:

Theorem C. (See Auscher [2].) *The Riesz transforms $\nabla^m L^{-1/2}$ are bounded operators on L^p if and only if $q_-(L) < p < q_+(L)$, where $q_{\pm}(L)$ denote the limits of p for the L^p boundedness of the family of operators $\{\sqrt{t} \nabla^m e^{-tL}\}_{t>0}$.*

Below we give the (H_L^1, L^1) boundedness of the Riesz transform $\nabla^m L^{-1/2}$.

Theorem 7.2. *The Riesz transforms $\nabla^m L^{-1/2}$ are bounded from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Proof. The theorem for $m = 1$ has been proved in [42, Theorem 3.3], in the following, we only consider the case of $m > 1$. By Theorem C and Theorem 7.1, we only need to prove that the operator $\nabla^m L^{-1/2}$ satisfies (7.1) and (7.2) for $p = 2$.

Let $E, F \subset \mathbb{R}^n$ be closed sets with $d(E, F) > 0$ and $f \in L^2(\mathbb{R}^n)$ with $\text{supp}(f) \subset E$. We first verify that the Riesz transform $\nabla^m L^{-1/2}$ satisfies (7.1). Note that $\nabla^m L^{-1/2}$ can be viewed as

$$\nabla^m L^{-\frac{1}{2}} f(x) = C \int_0^{\infty} \nabla^m e^{-sL} f(x) \frac{ds}{\sqrt{s}}.$$

Hence

$$\begin{aligned}
 \|\nabla^m L^{-\frac{1}{2}}(\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} &= C \left\| \int_0^\infty \nabla^m e^{-sL} (\mathbf{I} - e^{-tL})^M f \frac{ds}{\sqrt{s}} \right\|_{L^2(F)} \\
 &\leq C \int_0^\infty \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &\leq C \int_0^t \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &\quad + C \int_t^\infty \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &:= I_1 + I_2.
 \end{aligned} \tag{7.5}$$

By expanding $(\mathbf{I} - e^{-tL})^M$ by binomial formula and using (i) of Theorem 3.2, we have

$$\begin{aligned}
 I_1 &\leq C \int_0^t \|\nabla^m e^{-s(M+1)L} f\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &\quad + C \sup_{1 \leq k \leq M} \left(\int_0^t \|\nabla^m e^{-s(M+1)L} e^{-ktL} f\|_{L^2(F)} \frac{ds}{\sqrt{s}} \right) \\
 &\leq C \int_0^t e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right)^{2m/(2m-1)} \frac{ds}{s}} \|f\|_{L^2(E)} \\
 &\quad + C \sup_{1 \leq k \leq M} \left(\int_0^t \|\sqrt{kt} \nabla^m e^{-ktL} e^{-s(M+1)L} f\|_{L^2(F)} \frac{1}{\sqrt{kt}} \frac{ds}{\sqrt{s}} \right).
 \end{aligned}$$

And then for the second term of the last step, by Theorem 3.2 and Lemma 3.1 we get

$$\|\sqrt{kt} \nabla^m e^{-ktL} e^{-s(M+1)L} f\|_{L^2(F)} \leq C e^{-\left(\frac{d(E,F)}{ct^{1/2m}}\right)^{2m/(2m-1)}} \|f\|_{L^2(E)}.$$

Then for $M > \frac{n}{4m}(2m-1)$,

$$I_1 \leq C \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}. \tag{7.6}$$

For I_2 , we use (iii) of Theorem 3.2 for the operator $e^{-sL} - e^{-(t+s)L}$ and Lemma 3.1 to get

$$\begin{aligned}
 I_2 &= C \int_t^\infty \left\| \nabla^m e^{-sL} (e^{-sL} - e^{-(t+s)L})^M f \right\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &= C \int_t^\infty \left(\frac{t}{s} \right)^M \left\| \nabla^m e^{-sL} \left(\frac{s}{t} (e^{-sL} - e^{-(t+s)L}) \right)^M f \right\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &\leq C \int_t^\infty \left(\frac{t}{s} \right)^M e^{-\left(\frac{d(E,F)}{cs^{1/2m}} \right)^{2m/(2m-1)} \frac{ds}{s}} \|f\|_{L^2(E)} \\
 &\leq C \int_t^\infty \left(\frac{t}{s} \right)^M \left(\frac{s}{d(E,F)^{2m}} \right)^{\frac{M}{2m-1}} \frac{ds}{s} \|f\|_{L^2(E)} \\
 &\leq C \left(\frac{t}{d(E,F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}. \tag{7.7}
 \end{aligned}$$

Hence (7.1) holds for $\nabla^m L^{-1/2}$ by (7.5)–(7.13).

The verification for (7.2) is essentially similar. More precisely, one needs to estimate the integrals I_1 and I_2 in (7.5) with $(\mathbf{I} - e^{-tL})^M$ replaced by $(tLe^{-tL})^M$. As done before, by Lemma 3.1 we have

$$\begin{aligned}
 &\int_0^t \left\| \nabla^m e^{-s(M+1)L} (tLe^{-tL})^M f \right\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &\leq C \int_0^t \left\| \nabla^m e^{-tL} (tLe^{-tL})^{M-2} \left(\frac{t}{2} Le^{-\frac{t}{2}L} \right)^2 e^{-s(M+1)L} f \right\|_{L^2(F)} \frac{ds}{\sqrt{s}} \\
 &\leq C \int_0^t \left\| \sqrt{t} \nabla^m e^{-tL} (tLe^{-tL})^{M-2} \left(\frac{t}{2} Le^{-\frac{t}{2}L} \right)^2 e^{-s(M+1)L} f \right\|_{L^2(F)} \frac{1}{\sqrt{t}} \frac{ds}{\sqrt{s}} \\
 &\leq C e^{-\left(\frac{d(E,F)}{ct^{1/2m}} \right)^{2m/(2m-1)}} \int_0^t \frac{1}{\sqrt{t}} \frac{ds}{\sqrt{s}} \\
 &\leq C \left(\frac{t}{d(E,F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}. \tag{7.8}
 \end{aligned}$$

Concerning to the analogue of I_2 , we have

$$\int_t^\infty \left\| \nabla^m e^{-s(M+1)L} (tLe^{-tL})^M f \right\|_{L^2(F)} \frac{ds}{\sqrt{s}}$$

$$\begin{aligned}
&\leq C \int_t^\infty \left(\frac{t}{s}\right)^M \left\| \sqrt{s} \nabla^m e^{-sL} (sLe^{-(t+s)L})^M f \right\|_{L^2(F)} \frac{ds}{s} \\
&\leq C \int_t^\infty \left(\frac{t}{s}\right)^M e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right)2m/(2m-1)} \frac{ds}{s} \|f\|_{L^2(E)} \\
&\leq C \int_t^\infty \left(\frac{t}{s}\right)^M \left(\frac{s}{d(E,F)^{2m}}\right)^{\frac{M}{2m-1}} \frac{ds}{s} \|f\|_{L^2(E)} \\
&\leq C \left(\frac{t}{d(E,F)^{2m}}\right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}, \tag{7.9}
\end{aligned}$$

where we use (iii) of Theorem 3.2 for the operator $sLe^{-(t+s)L}$. Thus, (7.8) and (7.9) show that (7.2) holds also for $\nabla^m L^{-1/2}$. We therefore complete the proof of Theorem 7.2. \square

7.2. Vertical square functions

We have defined the vertical square functions $g_h^{L,k}$ and $g_p^{L,k}$ in Sections 3.3 and 6.2, respectively. Using the L^p boundedness of $g_h^{L,k}$ (Theorem 3.3) and $g_p^{L,k}$ (Theorem 6.2), we immediately get the following (H_L^1, L^1) boundedness of g_h^L and g_p^L by combining Lemma 3.3 (for $g_h^{L,k}$) and Lemma 6.2 (for $g_p^{L,k}$) with Theorem 7.1:

Theorem 7.3. *For $k \in \mathbb{N}$, the square functions $g_h^{L,k}$ and $g_p^{L,k}$ are all bounded operators from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Now, let us turn to two other vertical square functions $\tilde{g}_h^{L,m}$ and $\tilde{g}_p^{L,m}$, which associate to semigroups e^{-tL} and $e^{-t\sqrt{L}}$, respectively. For $f \in L^2(\mathbb{R}^n)$, denote

$$\tilde{g}_h^{L,m} f(x) := \left(\int_0^\infty |t^m \nabla^m e^{-t^{2m}L} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$\tilde{g}_p^{L,m} f(x) := \left(\int_0^\infty |t^m \nabla^m e^{-t^m\sqrt{L}} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Theorem 7.4. *The vertical square function $\tilde{g}_h^{L,m}$ can be extended to a bounded operator from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Proof. We first show that $\tilde{g}_h^{L,m}$ is bounded on $L^2(\mathbb{R}^n)$. Applying the idea used in the proof of (7.2) in [2], we have

$$\begin{aligned}
\|f\|_{L^2(\mathbb{R}^n)}^2 &= - \int_0^\infty \frac{d}{dt} \|e^{-tL} f\|_{L^2(\mathbb{R}^n)}^2 dt \\
&= \int_0^\infty \langle (L + L^*) e^{-tL} f, e^{-tL} f \rangle dt \\
&= 2\Re \iint_{\mathbb{R}_+^{n+1}} (Le^{-tL} f)(x) \overline{(e^{-tL} f)(x)} dx dt \\
&= 4m\Re \int_{\mathbb{R}_+^{n+1}} (s^m L e^{-s^{2m}L} f)(x) \overline{(s^m e^{-s^{2m}L} f)(x)} \frac{ds}{s} dx.
\end{aligned}$$

Then by the ellipticity assumptions (1.5) and (1.6) on L , we obtain that

$$\|f\|_{L^2(\mathbb{R}^n)} \approx \|\tilde{g}_h^{L,m} f\|_{L^2(\mathbb{R}^n)}. \quad (7.10)$$

To complete the proof of Theorem 7.4, by Theorem 7.1, it is sufficient to verify (7.1) and (7.2) for $\tilde{g}_h^{L,m}$ with $p = 2$. Let E and F be closed sets with $d(E, F) > 0$ and $f \in L^2(\mathbb{R}^n)$ with $\text{supp } f \subset E$. Write

$$\begin{aligned}
\|\tilde{g}_h^{L,m} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)} &= \left\| \left(\int_0^\infty |s^m \nabla^m e^{-s^{2m}L} (\mathbf{I} - e^{-tL})^M f|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^2(F)} \\
&\leq C \left(\int_0^\infty \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 ds \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 ds \right)^{\frac{1}{2}} \\
&\quad + C \left(\int_t^\infty \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 ds \right)^{\frac{1}{2}} \\
&:= I_1 + I_2. \quad (7.11)
\end{aligned}$$

By expanding $(\mathbf{I} - e^{-tL})^M$ by binomial formula and using (i) of Lemma 3.3, we have

$$\begin{aligned}
I_1 &\leq C \left(\int_0^t \|s^{1/2} \nabla^m e^{-s(M+1)L} f\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\
&\quad + C \sup_{1 \leq k \leq M} \left(\int_0^t \|\nabla^m e^{-s(M+1)L} e^{-ktL} f\|_{L^2(F)}^2 ds \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned} &\leq C \left(\int_0^t e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right)^{2m/(2m-1)} \frac{ds}{s}} \right)^{\frac{1}{2}} \|f\|_{L^2(E)} \\ &\quad + C \sup_{1 \leq k \leq M} \left(\int_0^t \|\sqrt{kt} \nabla^m e^{-ktL} e^{-s(M+1)L} f\|_{L^2(F)}^2 \frac{ds}{kt} \right)^{\frac{1}{2}}. \end{aligned}$$

Using the same argument as the one used to estimate I_1 in (7.5), we may get

$$I_1 \leq C \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}. \quad (7.12)$$

The idea to estimate I_2 is also the same with the one used to estimate I_2 in (7.5). Applying the conclusion (iii) of Lemma 3.3 for the operator $e^{-sL} - e^{-(t+s)L}$ and Lemma 3.1, we have for $m > 1$

$$\begin{aligned} I_2 &= C \left(\int_t^\infty \|\nabla^m e^{-s(M+1)L} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}^2 ds \right)^{\frac{1}{2}} \\ &= C \left(\int_t^\infty \left(\frac{t}{s} \right)^{2M} \left\| s^{1/2} \nabla^m e^{-sL} \left(\frac{s}{t} (e^{-sL} - e^{-(t+s)L}) \right)^M f \right\|_{L^2(F)}^2 \frac{ds}{s} \right)^{\frac{1}{2}} \\ &\leq C \left(\int_t^\infty \left(\frac{t}{s} \right)^{2M} e^{-\left(\frac{d(E,F)}{cs^{1/2m}}\right)^{2m/(2m-1)} \frac{ds}{s}} \right)^{\frac{1}{2}} \|f\|_{L^2(E)} \\ &\leq C \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}. \end{aligned} \quad (7.13)$$

When $m = 1$, by (7.13), we also have

$$\begin{aligned} I_2 &\leq C \left(\int_t^\infty \left(\frac{t}{s} \right)^{2M} e^{-\frac{d^2(E,F)}{cs} \frac{ds}{s}} \right)^{\frac{1}{2}} \|f\|_{L^2(E)} \\ &\leq C \left(\frac{t}{d(E, F)^2} \right)^M \|f\|_{L^2(E)}. \end{aligned}$$

To establish (7.2), one just needs to apply the same method as the one used to estimate the integrals I_1 and I_2 in (7.11) with $(\mathbf{I} - e^{-tL})^M$ replaced by $(tLe^{-tL})^M$ and the corresponding arguments used in proving Theorem 7.1. We omit the details here. Hence, the square function $\tilde{g}_h^{L,m}$ is a bounded operator from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. \square

Theorem 7.5. *The vertical square function $\tilde{g}_p^{L,m}$ can be extended to a bounded operators from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Proof. By the subordination formula (6.6), Minkowski's inequality and making the change of variables $s^{2m} := \frac{t^{2m}}{4z}$ we have

$$\begin{aligned}\tilde{g}_P^{L,m} f(x) &= C \left(\int_0^\infty \left| \int_0^\infty \frac{e^{-z}}{\sqrt{z}} t^m \nabla^m e^{-\frac{t^{2m}L}{4z}} f(x) dz \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq C \int_0^\infty e^{-z} z^{-\frac{1}{2} + \frac{1}{4}} \left(\int_0^\infty |s^m \nabla^m e^{-s^{2m}L} f(x)|^2 \frac{ds}{s} \right)^{\frac{1}{2}} dz \\ &\leq C \tilde{g}_h^{L,m} f(x).\end{aligned}\tag{7.14}$$

Thus, $\tilde{g}_P^{L,m}$ is a bounded operator on L^2 by (7.10) and (7.14). On the other hand, we have showed that, for closed sets E and F with $d(E, F) > 0$ and $f \in L^2(\mathbb{R}^n)$ with $\text{supp } f \subset E$,

$$\begin{aligned}&\left\{ \|g_h^{L,m} (\mathbf{I} - e^{-tL})^M f\|_{L^2(F)}, \|g_h^{L,m} (tLe^{-tL})^M f\|_{L^2(F)} \right\} \\ &\leq C \left(\frac{t}{d(E, F)^{2m}} \right)^{\frac{M}{2m-1}} \|f\|_{L^2(E)}.\end{aligned}\tag{7.15}$$

Using (7.14) again, we see that (7.15) also holds if replacing $g_h^{L,m}$ by $g_P^{L,m}$. We therefore get the (H_L^1, L^1) boundedness of the square function $\tilde{g}_P^{L,m}$ by Theorem 7.1. \square

7.3. Vertical maximal operators

For any $f \in L^2(\mathbb{R}^n)$, we consider the following vertical maximal operators

$$\mathcal{N}_h f(y) := \sup_{t>0} \left(\frac{1}{t^n} \int_{|x-y|<t} |e^{-t^{2m}L} f(x)|^2 dx \right)^{\frac{1}{2}}$$

and for $k \in \mathbb{N}$

$$\mathcal{N}_h^k f(y) := \sup_{t>0} \left(\frac{1}{t^n} \int_{|x-y|<t} |(t^{2m}L)^k e^{-t^{2m}L} f(x)|^2 dx \right)^{\frac{1}{2}}.$$

First we give the L^p boundedness of the vertical maximal operators \mathcal{N}_h and \mathcal{N}_h^k .

Theorem 7.6. *The operators \mathcal{N}_h and \mathcal{N}_h^k are both bounded on $L^p(\mathbb{R}^n)$ for $p_L < p \leq 2$.*

Proof. Since $p_L < p \leq 2$, we choose q such that $p_L < q < \min\{2, p\}$. For $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we use the L^q – L^2 off-diagonal estimate of the operator e^{-tL} (see (iv) in Theorem 3.2). Thus,

$$\begin{aligned}
\|\mathcal{N}_h f\|_{L^p(\mathbb{R}^n)}^p &\leq \int_{\mathbb{R}^n} \left[\sup_{t>0} \sum_{j=0}^{\infty} \left(\frac{1}{t^n} \int_{B(y,t)} |e^{-t^{2m}L}(\chi_{S_j(B(y,t)))} f)(x)|^2 dx \right)^{\frac{1}{2}} \right]^p dy \\
&\leq \int_{\mathbb{R}^n} \left[\sup_{t>0} \sum_{j=0}^{\infty} t^{-\frac{n}{2}} \|e^{-t^{2m}L}(\chi_{S_j(B(y,t)))} f)(x)\|_{L^2(B(y,t))} \right]^p dy \\
&\leq C \int_{\mathbb{R}^n} \left[\sup_{t>0} \sum_{j=0}^{\infty} t^{-\frac{n}{2}} t^{\left(\frac{n}{2}-\frac{n}{q}\right)} e^{-\left(\frac{d(B(y,t), S_j(B(y,t)))}{ct}\right)^{\frac{2m}{2m-1}}} \|f\|_{L^q(S_j(B(y,t)))} \right]^p dy \\
&\leq C \int_{\mathbb{R}^n} \left[\sup_{t>0} \sum_{j=0}^{\infty} t^{-\frac{n}{q}} 2^{-j\left(\frac{n}{q}+\varepsilon\right)} \|f\|_{L^q(S_j(B(y,t)))} \right]^p dy \\
&\leq C \int_{\mathbb{R}^n} \left[\sum_{j=0}^{\infty} 2^{-j\varepsilon} \sup_{t>0} (2^j t)^{-\frac{n}{q}} \|f\|_{L^q(S_j(B(y,t)))} \right]^p dy \\
&\leq C \int_{\mathbb{R}^n} (\mathcal{M}(|f|^q)(y))^{\frac{p}{q}} dy \\
&\leq C \|f\|_{L^p(\mathbb{R}^n)}^p,
\end{aligned}$$

where \mathcal{M} denotes the Hardy–Littlewood maximal operator and we use the $L^{\frac{p}{q}}$ boundedness of \mathcal{M} .

For the operator \mathcal{N}_h^k , we just use the L^q – L^2 off-diagonal estimate of the operator tLe^{-tL} (see (iv) in Theorem 3.2) in the previous argument, then the same method can be applied to get the desired L^p boundedness of \mathcal{N}_h^k . \square

Theorem 7.7. *The operator \mathcal{N}_h is bounded from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Proof. By Lemma 5.1, we only need to show that for every $(2, \varepsilon, M)$ -molecule u associated to some cube Q

$$\|\mathcal{N}_h u\|_{L^1(\mathbb{R}^n)} \leq C. \quad (7.16)$$

By Hölder’s inequality and annular decomposition of \mathbb{R}^n we have

$$\begin{aligned}
\|\mathcal{N}_h u\|_{L^1(\mathbb{R}^n)} &\leq \sum_{j=0}^{\infty} |S_j(Q)|^{\frac{1}{2}} \|\mathcal{N}_h u\|_{L^2(S_j(Q))} \\
&\leq \sum_{j=0}^{10} |S_j(Q)|^{\frac{1}{2}} \|\mathcal{N}_h u\|_{L^2(S_j(Q))} + \sum_{j=11}^{\infty} |S_j(Q)|^{\frac{1}{2}} \|\mathcal{N}_h u\|_{L^2(S_j(Q))} \\
&:= I_1 + I_2.
\end{aligned} \quad (7.17)$$

For I_1 , by the $L^2(\mathbb{R}^n)$ boundedness of \mathcal{N}_h and the definition of a $(2, \varepsilon, M)$ -molecule, we have

$$I_1 \leq \sum_{j=0}^{10} |S_j(Q)|^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} \leq C.$$

To deal with I_2 , we fix some constant $0 < a < 1$ such that $aM - \frac{n}{4m}(2m-1) > 0$ and split $\mathcal{N}_h u$ into two parts with $t < 2^{aj}l(Q)$ and $t \geq 2^{aj}l(Q)$.

(a) Case where $t < 2^{aj}l(Q)$. Let

$$U_j(Q) = 2^{j+3}Q \setminus 2^{j-3}Q, \quad R_j(Q) = 2^{j+5}Q \setminus 2^{j-5}Q, \quad E_j(Q) = R_j^c(Q),$$

and split $u = u\chi_{R_j(Q)} + u\chi_{E_j(Q)}$. Then for $y \in S_j(Q)$ and $|x - y| < t$, we have $x \in U_j(Q)$. Hence, by (i) of Theorem 3.2

$$\begin{aligned} \left(\frac{1}{t^n} \int_{|x-y|<t} |e^{-t^{2m}L}(u\chi_{E_j(Q)})(x)|^2 dx \right)^{\frac{1}{2}} &\leq C t^{-\frac{n}{2}} e^{-\left(\frac{d(E_j(Q), U_j(Q))}{t}\right)^{\frac{2m}{2m-1}}} \|u\|_{L^2(E_j(Q))} \\ &\leq C t^{-\frac{n}{2}} \left(\frac{t}{2^j l(Q)} \right)^{\frac{2mN}{2m-1}} \|u\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where we choose $N \in \mathbb{N}$ satisfying $N > \frac{2-a}{1-a} \frac{n}{4m}(2m-1)$. Thus, we get

$$\begin{aligned} &\sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} \left\| \sup_{t < 2^{aj}l(Q)} \left(\frac{1}{t^n} \int_{|x-y|<t} |e^{-t^{2m}L}(u\chi_{E_j(Q)})(x)|^2 dx \right)^{\frac{1}{2}} \right\|_{L^2(S_j(Q))} \\ &\leq C \sum_{j=10}^{\infty} 2^{j(n(1-\frac{a}{2}) + \frac{2mN}{2m-1}(a-1))} \leq C. \end{aligned} \quad (7.18)$$

On the other hand, by the $L^2(\mathbb{R}^n)$ boundedness of \mathcal{N}_h and the definition of a $(2, \varepsilon, M)$ -molecule we have

$$\begin{aligned} &\sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} \left\| \sup_{t < 2^{aj}l(Q)} \left(\frac{1}{t^n} \int_{|x-y|<t} |e^{-t^{2m}L}(u\chi_{R_j(Q)})(x)|^2 dx \right)^{\frac{1}{2}} \right\|_{L^2(S_j(Q))} \\ &\leq C \sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} \|\mathcal{N}_h(u\chi_{R_j(Q)})\|_{L^2(S_j(Q))} \\ &\leq C \sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} \|u\|_{L^2(R_j(Q))} \leq C. \end{aligned} \quad (7.19)$$

(b) Case where $t \geq 2^{aj}l(Q)$. In this case, for every $y \in \mathbb{R}^n$

$$\begin{aligned}
 & \sup_{t \geq 2^{aj}l(Q)} \left(\frac{1}{t^n} \int_{|x-y|<t} |e^{-t^{2m}L}u(x)|^2 dx \right)^{\frac{1}{2}} \\
 &= \sup_{t \geq 2^{aj}l(Q)} \left(\frac{1}{t^n} \int_{|x-y|<t} |(t^{2m}L)^M e^{-t^{2m}L}((t^{2m}L)^{-M}u)(x)|^2 dx \right)^{\frac{1}{2}} \\
 &\leq \sup_{t \geq 2^{aj}l(Q)} \left(\frac{1}{t^n} \int_{|x-y|<t} |(t^{2m}L)^M e^{-t^{2m}L}((2^{2maj}l(Q)^{2m}L)^{-M}u)(x)|^2 dx \right)^{\frac{1}{2}} \\
 &\leq C 2^{-2Mmaj} \mathcal{N}_h^M((l(Q)^{2m}L)^{-M}u)(y). \tag{7.20}
 \end{aligned}$$

Then by the $L^2(\mathbb{R}^n)$ boundedness of \mathcal{N}_h^M (Theorem 7.6) and the fact that $aM - \frac{n}{4m}(2m-1) > 0$, we have

$$\begin{aligned}
 & \sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} \left\| \sup_{t \geq 2^{aj}l(Q)} \left(\frac{1}{t^n} \int_{|x-y|<t} |e^{-t^{2m}L}u(x)|^2 dx \right)^{\frac{1}{2}} \right\|_{L^2(S_j(Q))} \\
 &\leq C \sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} 2^{-2Mmaj} \|\mathcal{N}_h^M((l(Q)^{2m}L)^{-M}u)\|_{L^2(R_j(Q))} \\
 &\leq C \sum_{j=10}^{\infty} |S_j(Q)|^{\frac{1}{2}} 2^{-2Mmaj} \|(l(Q)^{2m}L)^{-M}u\|_{L^2(\mathbb{R}^n)} \\
 &\leq C \sum_{j=10}^{\infty} 2^{-j(2Mma-\frac{n}{2})} \leq C. \tag{7.21}
 \end{aligned}$$

Thus, (7.16) follows by (7.17)–(7.21). \square

8. Final remarks

Remark 8.1. It is well known that the classical Hardy space $H^1(\mathbb{R}^n)$ associated with the operator $L \equiv -\Delta$, the Laplacian, can be characterized by the area integral, the vertical square function, the vertical maximal function and the Riesz transforms.

If $L = \mathcal{P}$, the homogeneous elliptic operator of order $2m$ ($m \geq 1$) with real constant coefficients (see (1.3)), then Theorems 2.1–2.3 of the present paper show that the following equivalent relationships hold: for $f \in L^1(\mathbb{R}^n)$

$$\begin{aligned}
 \|f\|_{H^1(\mathbb{R}^n)} &\approx \|\mathcal{S}^{\mathcal{P}}f\|_{L^1} \approx \|g^{\mathcal{P}}f\|_{L^1} \approx \|\mathcal{N}^{\mathcal{P}}f\|_{L^1} \\
 &\approx \|\mathcal{S}^{\sqrt{\mathcal{P}}}f\|_{L^1} \approx \|g^{\sqrt{\mathcal{P}}}f\|_{L^1} \approx \|\mathcal{N}^{\sqrt{\mathcal{P}}}f\|_{L^1} \\
 &\approx \|\nabla^m L^{-\frac{1}{2}}f\|_{L^1} \quad (m \text{ is an odd integer}).
 \end{aligned}$$

When L is a second order elliptic operator in divergence form with complex bounded measurable coefficients, in their nice paper [42], Hofmann and Mayboroda proved that

$$\|f\|_{H_L^1} \approx \|S_h^L f\|_{L^1} \approx \|S_p^L f\|_{L^1} \approx \|\mathcal{N}_h f\|_{L^1} \approx \|\mathcal{N}_p f\|_{L^1}.$$

However, for a general higher order elliptic operator L with variable coefficients, it seems to be a difficult problem to characterize the Hardy space $H_L^1(\mathbb{R}^n)$ by the vertical square function or the vertical maximal functions associated with the operator L . Theorem 2.3 of the present paper indeed shows that this problem becomes possibly more complex for the case of the higher order elliptic operator. In fact, up to now, to our knowledge, for the higher order divergence form elliptic operator L discussed in the present paper (see (1.4)–(1.6)), it is still an open question that whether the following equivalent relationships hold:

$$\begin{aligned} \|f\|_{H_L^1} &\approx \|g_h^L f\|_{L^1} \approx \|g_p^L f\|_{L^1} \approx \|\tilde{g}_h^L f\|_{L^1} \approx \|\tilde{g}_p^L f\|_{L^1} \approx \|\mathcal{N}_h f\|_{L^1} \\ &\approx \|\mathcal{N}_p f\|_{L^1} \approx \|\nabla^m L^{-\frac{1}{2}} f\|_{L^1}. \end{aligned}$$

Remark 8.2. It would be very interesting to further establish a similar theory as done here for generalized Schrödinger operator $(-\Delta)^m + V$ where $m \geq 2$ and V is a measurable function with maybe some singularities. However, there exist some obstacles to follow usual techniques for classical Schrödinger operator (see e.g. [33,30–32]).

Firstly, note that $e^{-t(-\Delta)^m}$ is not a preserving-positivity semigroup and also not a contractive one on $L^p(\mathbb{R}^n)$ ($p \neq 2$) for $m \geq 2$ (see, e.g. Reed and Simon [56], Langer and Maz'ya [48]). Thus, it would make rather difficult to use the famous Trotter formula of semigroup and as well lose fascinating connection to Brownian motion by Feynmann–Kac formula, which both are essential tools in many problems of classical Schrödinger operators (see, e.g. Davies [17], Simon [57]).

Secondly, the higher order elliptic operator $(-\Delta)^m$ ($m \geq 2$) also lacks some important properties such as maximum principle which is basic to second order elliptic operator (see Gilbarg and Trudinger [38] and Gazzola, Grunau and Sweers [37]).

In spite of these crucial difficulties, nevertheless, there exist many interesting works devoted to the L^p -theory of generalized Schrödinger operator, for instance, [23,24,64] and so on. In our recent work [26], some useful L^p – L^q off-diagonal estimates of the higher order Schrödinger semigroup $e^{-t((-\Delta)^m + V)}$ ($m \geq 2$) were firstly established for a class of singular potentials V . As an application of these estimates, we have studied the L^p boundedness of Riesz transforms associated to the higher order Schrödinger type operator $(-\Delta)^m + V$ for $m \geq 2$ and this kind of potentials V . In particular, it should be emphasized that the specific singular potential V discussed in [26] does not need to be a nonnegative function.

Finally, we would like to point out that for this class of singular potentials V appearing in [26], a similar Hardy space theory associated to generalized Schrödinger operator $(-\Delta)^m + V$ as $m \geq 2$ also can be finally constructed as done in the present paper. These progresses will be stated in our forthcoming papers.

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